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ACOUSTICS

THEORETICAL

PART I

BY

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ADVERTISEMENT.

As this is the only portion of a treatise on Acoustics, intended to comprise the practical as well as the theoretical parts of the subject, which will proceed from the pen of its Author, a few words are required to explain the circumstances under which it now appears.

The Author, the late Professor Donkin, has passed away prematurely from the work. It was a work he was peculiarly qualified to undertake, being a mathematician of great attainments and rare taste, and taking an especial interest in the investigation and application of the higher theorems of analysis which are necessary for these subjects. He was, moreover, an accomplished musician, and had a profound theoretical knowledge of the Science of Music.

He began this work early in the year 1867; but he was continually interrupted by severe illness, and was much hindered by the difficulty, and in many instances the impossibility, of obtaining accurate experimental results at the places wherein his delicate health compelled him to spend the winter months of that and the following years. He took, however, so great an interest in the subject, that he continued working at it to within two or three days of his death.

The part now published contains an inquiry into the Vibrations of Strings and Rods, together with an explanation of the more elementary theorems of the subject, and is, in the opinion of its Author, complete in itself; his wish was that it should be published as soon as possible; and he was pleased at knowing that the last pages of it were passing through the Press immediately before the time of his death. It is the first portion of the theoretical part.

It was intended that the second portion should contain the investigation into the Vibrations of Stretched Membranes and Plates; into the Motion of the Molecules of an Elastic Body; and into the Mathematical Theory of Sound. Professor Donkin did not live long enough to complete any part of this section of the work.

The third portion was intended to contain the practical part of the subject; and the theory and practice of Music would have been most fully considered. It is exceedingly to be regretted that the Professor did not live to complete this portion; for the combination of the qualities necessary for it is seldom met with, and he possessed them in a remarkable degree. Not even a sketch or an outline is found amongst his papers. He had formed the plan in his own mind and often talked of it with pleasure. It can now never be written as he would have written it.

BARTHOLOMEW PRICE.

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CHAPTER I.

GENERAL INTRODUCTION.

1. The sensation of sound, like that of light, may be produced in exceptional and extraordinary ways. But the first step in the usual process consists in the communication of a vibratory motion to the tympanic membrane of the ear, through slight and rapid changes in the pressure of the air on its outer surface.

The ear may be considered as consisting essentially of two parts, of which one is the organ of communication with the external world, and the other of communication with the brain.

2. The former part is a tube of irregular form, divided into two portions of nearly equal length by the tympanic membrane, which is stretched across it somewhat obliquely as a transverse diaphragm. The shorter and wider part of the tube is outside the tympanic membrane, and ends at the orifice of the external ear. This part is called the Meatus.

The part of the tube immediately within the tympanic membrane is called the Tympanum, and the remainder the Eustachian Tube.

The Eustachian tube leads into the *pharynx*, that is, the cavity behind the tonsils and uvula, into which the nostrils also open. But the orifice of the Eustachian tube, though capable of being opened, is usually closed. It is opened involuntarily in the act of swallowing, and can be opened voluntarily by a muscular effort not easily described but easily made; and the opening is accompanied by a slight sensation in the ear, due to a temporary change in the pressure of the air in the tympanum. Thus the meatus on the one hand, and the tympanum and

Eustachian tube on the other, always contain air, but under different conditions. The air in the meatus is liable to be directly affected by every change, however slight and rapid, in the pressure of the external air, with which it is always in free communication; but that within the tympanic membrane, being only occasionally put into communication with the external air by the opening of the Eustachian tube, is not liable to be directly affected by slight and rapid changes, though it takes part in the slower fluctuations shewn by the barometer.

3. The second, or interior, part of the ear is contained within a cavity which is called the *bony labyrinth*, because it is of a complicated form, and is surrounded by bone except in two places. These two places may be compared to windows, looking into the tympanum, but completely closed by membranes, so that neither air nor fluid can pass through them. One of these is called the *oval* and the other the *round* window.

The interior of this bony labyrinth is filled with fluid, in which are suspended membranous bags, following nearly the same form, and themselves containing fluid.

The terminal fibres of the auditory nerve are distributed over the surfaces (or parts of the surfaces) of these membranous bags, and there are special arrangements of which the object appears to be the communication to these nervous fibres of any agitation affecting the fluid.

4. The tympanic membrane is connected with that which closes the 'oval window' by a link-work of small bones contained in the open space of the tympanum, in such a manner that when the former membrane is bulged inwards or outwards by an increase or diminution of the pressure on its external surface, a similar movement is impressed on the latter; and although the fluid within is probably as incompressible as water (of which it chiefly consists), the membrane of the 'round window' allows it to yield by expanding outwards when that of the oval window is forced inwards, and *vice versâ*.

Thus the motion impressed on the tympanic membrane by the external air is communicated to the fluid contained in the labyrinth, and from that to the fibres of the auditory nerve, by means of the apparatus mentioned above, which need not be further described at present[1].

It is probable also that motion is partly propagated from the tympanic membrane through the air in the tympanum to the membrane of the round window, and so to the fluid.

5. The 'pressure' of the air at any point must be understood to mean, as usual, the pressure which would be exerted on a unit of surface by air of the same density and temperature as at the point in question.

When the pressure at any point varies with the time, the variation may be graphically represented by means of a 'curve

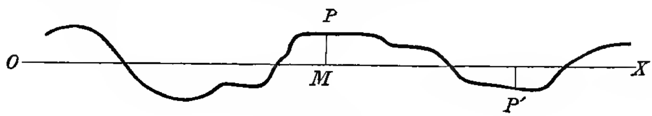

Fig. 1.

of pressure,' in which the abscissa OM of any point P is proportional to the time elapsed since a given instant, and the ordinate MP to the excess of pressure above a standard value, which may be taken arbitrarily. A negative ordinate (as at P') represents of course a defect of pressure below the standard value.

As we shall chiefly have occasion to consider cases in which the *average* (or mean) pressure remains unaltered, it will be convenient to assume that average as the standard value represented by the axis of abscissæ OX.

Changes of density may evidently be represented in the same way by a 'curve of density,' in which positive ordinates represent condensation, and negative ordinates dilatation. The curves of pressure and of density will differ slightly in form, because

[1] A full description, with illustrations, of the structure of the ear, so far as it is known at present, will be found in Helmholtz, p. 198, &c., and in Huxley's 'Lessons in Elementary Physiology,' p. 204, &c., and other recent works on Anatomy and Physiology. But the reader is recommended to study the subject, if possible, with the help of anatomical preparations or models. For the purposes of this treatise, however, nothing is absolutely necessary to be known beyond what is stated here or hereafter in the text.

pressure is not in general strictly proportional to density; the reasons and consequences of this fact, however, do not concern us at present.

Although it is convenient to use the word 'curve,' it must be understood that the lines representing changes of pressure or density are not necessarily curved in the ordinary sense, but may consist either wholly or in part of straight portions.

6. Those slight and rapid changes in the pressure of the air in contact with the tympanic membrane, which cause (Art. **1**) the sensation of sound, do not in general alter the average pressure; so that they would be represented by a wavy curve including upon the whole equal areas above and below the axis of abscissæ.

A wavy curve may or may not be *periodic*. A periodic curve consists of repetitions of a single portion, thus:

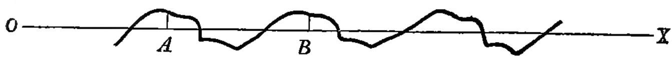

Fig. 2.

The *period* or *wave-length* of such a curve is the smallest distance $A\ B$ which, measured from any arbitrary point along the axis, has the ordinates at its extremities always equal: in other words, it is the projection, on the axis, of the smallest portion which is repeated.

7. Sounds are usually divided into two classes: namely, unmusical sounds, or *noises;* and musical sounds, or *notes*.

As regards sensation, the distinction between these two classes of sounds is said to be that notes have *pitch* and noises have not; and, as regards the mode of their production, that the curve of pressure (Art. **6**) is periodic in the case of a musical sound, and non-periodic in the case of a noise. But these statements require more explanation and correction than might at first sight be expected.

In the first place, it is obvious to common observation that few, if any, noises are perfectly unmusical, that is, absolutely without pitch. Two noises of the same general character often differ from one another in a way which we describe by calling one of them more acute or sharp, and the other more grave or flat; for example, the reports of a pistol and of a cannon.

On the other hand, few, if any, sounds are perfectly musical, that is, absolutely unmixed with noise.

Hence the question presents itself whether there is after all a real distinction in kind between noises and notes, and if there is, in what it consists.

8. Before attempting to answer this question, we must notice two facts of fundamental importance.

The first is, that any sound whatever, if repeated at equal and sufficiently short intervals of time, generates a *note*, of which the pitch depends upon the frequency of repetition of the original or elementary sound. This is easily verified by simple experiments. For instance, if the point of a quill pen, or the edge of a card, be held against the teeth of a wheel which is turned slowly, the passage of each tooth produces a sharp noise or 'click.' But if the velocity of rotation of the wheel be gradually increased, the clicks gradually cease to be heard separately, and are replaced by a sound which gradually acquires a continuous character, and a pitch which rises as the velocity increases. The connection between the frequency of repetition of the elementary sound, and the pitch of the resultant note, will be considered afterwards.

The second fact is, that more than one sound can be heard at once. This is familiar to every one. But the following considerations will shew that it is very remarkable.

9. In the case above supposed, each passage of a tooth of the wheel across the quill or card produces a disturbance in the air, which is propagated in all directions (unless some obstacle intervene) in the form of a *wave*.

The nature of sound-waves in air will be considered hereafter. At present it is sufficient to say that though they are essentially different in most respects from ordinary waves in water, they have one important property in common, namely, that different sets of waves can be propagated at the same time, either in the same or in different directions, without destroying one another. And it results from the mechanical theory that when the waves are very small, the different sets are simply *superposed;* that is, the disturbance produced at any point, and at any given time, by the combined action of waves belonging to different sets, is the sum of the disturbances which would have been produced by the waves of each set separately.

In this proposition the word *sum* is to be understood, according to circumstances, either in the ordinary algebraical sense, or in the extended sense in which the diagonal of a parallelogram is called (in symbolical geometry) the *sum* of two contiguous sides.

Thus, in the case of sound-waves in air, the disturbance at any point may be considered either as a displacement of particles, or as an alteration of pressure and density. And the whole change, either of pressure or density, is the algebraical sum of the changes which would have been produced by waves of each set separately; while the displacement of any particle is the *sum* of the separate displacements in the sense just explained.

Thus, suppose A is the undisturbed position of a particle which, at a given instant, would be displaced to P by the action of a wave belonging to one set, and to Q by that of a wave belonging to another set. Then the actual displacement at the same instant will be to R, AR being the diagonal of the parallelogram constructed upon AP, AQ.

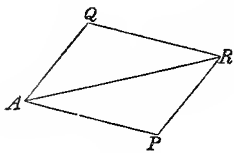

Fig. 3.

This law of the composition of displacements, which is identical in form with that of the composition of forces in Mechanics, may be stated in another manner thus:

Each separate cause of displacement acts independently of other causes; a proposition which is to be understood as follows:—

The actual displacement AR is the same as if the particle were first displaced by one cause along AP, and then by the other along a line PR, *equal and parallel* to the displacement

A Q, which the latter cause would have produced if acting alone.

It is important to recollect that this law of the 'superposition of displacements' is not a universal law in the same sense as that of the composition of forces. It depends, in fact, upon the condition that the force which tends to restore a displaced particle to its undisturbed position is directly proportional to the displacement; and this condition, in most cases in which it subsists at all, subsists rigorously only for *infinitely small* displacements.

The law, however, is sensibly true, so far as most of the phænomena of sound are concerned, in the case of the greatest disturbances produced by ordinary causes.

10. This being premised, let us consider a continuous noise, lasting say for a small fraction of a second, in which the ear can recognise no definite pitch. And suppose the curve of

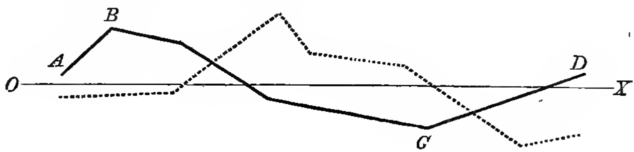

Fig. 4.

pressure (that is, the curve representing in the way above explained (Art. **5**) the changes of pressure close to the tympanic membrane) to be the black line *A B C D*, Fig. 4.

Suppose a different noise, of the same duration, to have for its curve of pressure the dotted line in the same figure. Then, if

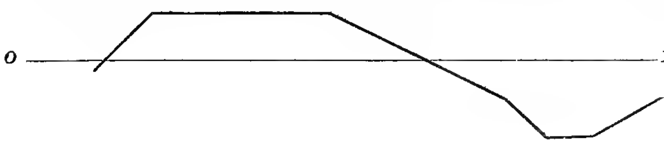

Fig. 5.

the causes producing such noises both act at once, the resultant variation of pressure will be represented, as in Fig. 5, by a curve, of which the ordinates are the algebraical sums of the corresponding ordinates in Fig. 4.

Now in such a case the two noises do not in general coalesce into one, but are heard distinctly, though simultaneously. It is only when the two curves of Fig. 4 are nearly similar, that is, when the two noises are very like one another, that the ear does not easily resolve the resultant sound into its two components.

On the other hand, there is nothing in the curve of Fig. 5, which can suggest to the eye the process of composition by which it was generated. It looks quite as simple as either of its components; and although it might be arbitrarily resolved into two in an infinite variety of ways, there is nothing in its appearance to indicate one way as more natural than any other[1].

It is evident then that the ear has a power of resolution which the eye has not; or rather, that the ear resolves according to some law peculiar to itself, whereas the eye either does not resolve at all, or resolves arbitrarily.

11. This phænomenon is still more remarkable when the component curves are periodic. Let *A B C D E F G*, Fig. 6,

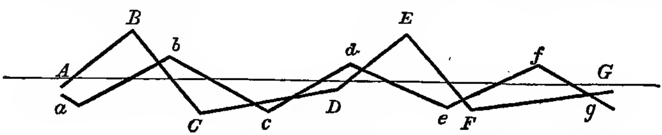

Fig. 6.

represent two periods of a periodic change of pressure; the curve consisting of repetitions of the portion *A B C D*. Similarly let *a b c d e f g* represent *three* periods of another periodic change, occupying the same time as two periods of the former.

[1] The curves in the above figures are represented as made up of straight lines, merely for convenience of drawing. Whether their forms be or be not such as could really occur, is quite immaterial to the argument. The point essential to be understood is this: that whatever be the forms of the component curves, the eye cannot in general distinguish them in the resultant curve.

The resultant curve consists of repetitions of the portion $P\,Q$, Fig. 7, and is therefore also periodic.

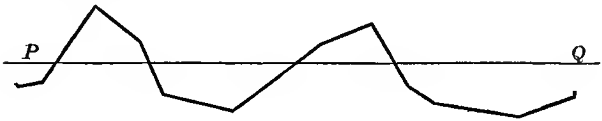

Fig. 7.

If the periods be sufficiently short, each of the curves in Fig. 6 will correspond to a musical note; and the curve in Fig. 7 represents the variation of pressure which takes place when the two notes are heard simultaneously. Now, it is a well-known fact that in such a case the ear in general distinguishes both notes; that is, it resolves the resultant curve of Fig. 7 into the two components of Fig. 6. But the eye sees nothing more in Fig. 7 than a curve having a period represented by the abscissa $P\,Q$, and entirely fails to distinguish the different periods of the component curves. Here again, then, we find that the ear resolves according to a law of its own.

12. But a new question now presents itself. If a curve like $P\,Q$ is resolved by the ear into two components, why is not each component similarly resolved? If the repetition of $P\,Q$ represents two simultaneous but distinguishable notes, why may not the repetition of $A\,B\,C\,D$ represent two notes in the same way; and why may not each component of $A\,B\,C\,D$ be itself again resolved, and so on *ad infinitum?*

The answer is, that all this does in general really happen; and that a perfectly simple musical tone, that is, a tone such as the ear cannot resolve, is rarely heard except when produced by means specially contrived for the purpose.

Thus the sound produced by a vibrating string, of a pianoforte or violin for instance, is in general compounded of simple tones, theoretically unlimited in number. Only a few of them, however, are loud enough to be actually heard. These few constitute a combination which is *always* heard from the same string under the same circumstances; hence we acquire a habit of associating them together and perceiving them as a single note of a special character; and it requires an effort of attention, which is difficult when first attempted by the unassisted ear, to analyse the compound sensation. In the case of a string this difficulty is increased by the circumstance that the audible component tones form in general an agreeable consonance. When, however, two strings, not in unison, vibrate at once, we distinguish their notes perfectly; partly because there is in this case no habit of association, and partly because the component tones of one do not in general form a consonant combination with those of the other.

The sound of a large bell is compounded of many simple tones, some of which combine agreeably and some produce harsh dissonances. In this case every one perceives easily the complex character of the sound. Still the habit of association prevents us from mistaking the sound of one bell for the sound of two; and on the other hand, when two bells are struck at once, we distinguish the two compound sounds more or less perfectly; notwithstanding the very confused combination of simple tones.

13. We can now give at least a partial answer to the question, What is a noise? It is in general a combination of a number of musical tones too near to one another in pitch to be distinguished by the unassisted ear.

The effect produced by ringing all the bells of a peal at once, or striking all the twelve keys of an octave on a pianoforte at once, shews how a confused combination of tones tends to become a noise. And it may be easily conceived that the change would be much more complete if, for example, twelve notes intermediate between C and C♯ were heard at once.

It appears, then, that a noise and a simple tone are extreme cases of sound. The former is so complex that the ordinary powers of the ear fail to resolve it. The latter is incapable of resolution by reason of its absolute simplicity.

14. It is evident therefore that the question, *What is an absolutely simple tone?* or rather, What is the character of the sound-waves which produce such tones, and of the corresponding curves of pressure? is of fundamental importance.

It is evident also that some correction is required to the statement referred to above (Art. 7), that musical notes have *pitch*. Strictly speaking, only an absolutely simple tone has a single determinate pitch. When we speak of the pitch of the note produced by a string or an organ-pipe, we mean in fact the pitch of the *gravest* simple tone in the combination. This particular component is in general louder than any of the others, and is that on which attention is fixed. The others become associated with it by habit, and seem only to modify its character, without destroying the unity of its pitch.

15. As it will often be necessary to distinguish between simple and compound musical sounds, we shall always use (as in the preceding articles) the words *tone* and *note* for this purpose.

Objections may easily be made to both these terms. In fact the word *tone* is often used with express reference to that very complexity which it is here intended to exclude; as when we say that the tone of a violin is different from that of a clarinet, or that any instrument has a good or bad tone. Again, *note* properly signifies the written mark which indicates what musical sound is to be played or sung, and not the sound itself.

But it may be answered that *tone* (Gr. τόνος) really means *tension*, and the effect of tension is to determine the *pitch* of the sound of a string. Hence the word may naturally be used to denote a sound with reference only to its pitch; and therefore, in particular, to denote a simple sound which has a single pitch, and has (as will be seen hereafter) no other distinctive quality except loudness or softness.

And with respect to *note*, it may be answered that the transition from the written mark to the thing signified is in fact habitually made, as when we say that a person sings wrong notes.

Again, the written note is a direction to sing or play, not a simple sound, but that particular complex sound which is produced by the voice or by the instrument intended to be used; and the particular sound produced is in fact a *note* or characteristic mark of the kind of instrument. In this sense we speak of the note of the blackbird or of the nightingale.

16. For these reasons we shall henceforth denote a simple musical sound by the word *tone*, and the elementary sound of any instrument, such as that of a violin string, an organ-pipe, or the human voice, by the word *note*.

Thus a note will be in general a complex sound, though it may accidentally, through the peculiarity of a particular instrument, be simple, or nearly so [1].

[1] Helmholtz uses the words *Klang* and *Ton* to signify compound and simple musical sounds. We have followed him in adopting the latter term. But such a sound as that of the human voice could hardly be called in English a *clang*, without doing too much violence to established usage.

CHAPTER II.

MISCELLANEOUS DEFINITIONS AND PROPOSITIONS.

THE contents of this Chapter will consist chiefly of definitions and statements of fact, which may conveniently be introduced here, though partly belonging to a later stage of the subject.

17. The word *vibration* may be used to denote *any periodic change of condition;* especially when, relatively to appropriate standards of comparison, the change is small, and the period short.

When a change is called periodic, it is in general implied that a period is an interval of time of constant length, and that whatever condition exists at any instant, is restored after the lapse of a period.

Thus the series of changes which happen between any two instants separated by a period, constitute a cycle which is continually repeated.

The particular stage of change which has been reached at any instant is called the *phase* of the vibration. And the vibration is said to go through all its phases in one period.

The period is also called the *time of a vibration.* It may, and frequently does happen, that the changes in successive periods grow successively less, though remaining similar in character, while the period continues sensibly unaltered. Thus when a string is put into a state of vibration and then left to itself, the sound gradually dies away, but retains sensibly both the same pitch and the same character as long as it is heard at all. On the other hand, the period may change, as when the tension of a string is increased or diminished while it is vibrating.

But when nothing is said to the contrary, it is always to be understood that the period is supposed to be constant.

18. When a musical note (Art. **16**) is heard in the ordinary way, the air in contact with the tympanic membrane of the ear is in a state of vibration. But it has been already seen that the ear in general resolves a series of vibrations into several component series having different periods, each of which produces a simple tone with a determinate pitch. The tone produced by the component vibrations of longest period, being the lowest in pitch, and also in general the loudest, is called the *fundamental tone*, and its pitch is usually spoken of as the pitch of the note.

We shall, however, suppose at present that the vibrations are of that kind which the ear does not resolve, so that only one tone is heard; and proceed to state certain facts relative to the connection between the pitch of the tone and the period of the vibrations.

To avoid circumlocution, we may call the vibrations which produce any particular tone the *vibrations of the tone;* and the period of the vibrations may be called the *period of the tone.*

19. If the period of the vibrations be too long, no tone is perceived, but only a succession of distinct impulses which affect the ear with a peculiar sensation different both from noise and tone. If, on the other hand, the period be too short, no sensation at all is produced; the sound is simply inaudible.

Different observers have made various statements as to the period of the slowest vibrations which produce a tone. But it is certain that to all ordinary ears the perception of pitch begins when the number of vibrations in a second is somewhere between eight and thirty-two. In experiments on this point there is an uncertainty arising from the doubt whether the tone heard is really fundamental, or only one of the component tones having a shorter period than that of the compound vibration. For it is certainly difficult and perhaps impossible so to arrange the experiment as to be quite sure that the vibration is of that kind which the ear does not resolve.

The extreme limit in the contrary direction is known to be different to different ears[1]. In general it is probable that no tone is heard when the number of vibrations in a second exceeds 40,000.

20. Difference of period causes a difference of pitch which we describe by calling tones of shorter period higher, or more acute, and those of longer period lower, or more grave.

The change of pitch which takes place when the period is gradually altered, strikes us as having an analogy to the gradual change of position of a moving point. Hence we say that there is an *interval* between tones of different pitch.

Suppose P, Q, R are three tones, taken in order of pitch. Then we say that the interval from P to R is the sum of the intervals from P to Q and from Q to R; or briefly, that $PR = PQ + QR$.

So far as this we might go in many other cases of continuously varying sensation. For instance, we might speak of the interval between two pains of the same kind, but of different intensities, and call one interval the sum of two others. But in the comparison of tones we can make a further step which we cannot make in the comparison of pains. For we can compare with precision the *magnitudes of two intervals*. The ear decides whether any interval PQ is greater than, equal to, or less than any other interval RS. How far this faculty of comparing intervals is an absolutely simple and ultimate property of the sense of hearing, it is not very easy to decide, for reasons which will appear hereafter.

21. But however this may be, it is certain that the judgment is made; and it is a fact ascertained by experiment that any interval PQ is judged to be equal to another interval RS, whenever the periods (Art. **18**) of the two tones P, Q, are to

[1] This appears to have been first observed by Wollaston. See his memoir 'On Sounds inaudible by certain Ears,' *Phil. Trans. for* 1820, p. 306. On the audibility of high tones, see Savart in *Ann. de Chim. et de Phys.*, t. 44, p. 337; on low tones, Helmholtz, p. 263.

one another in the same ratio as the periods of the tones R, S. The same proposition may of course be expressed by saying that the intervals are equal when the ratios of the numbers of vibrations in a given time are equal; for the periods are inversely proportional to the numbers of vibrations. Thus, if P, Q be produced by 200 and 300 vibrations in a second, and R, S by 600 and 900, then the intervals PQ, RS are equal, for 200 : 300 :: 600 : 900.

22. In general, if a tone P be produced by p vibrations in a second, and another tone Q by q vibrations, the ratio $\frac{q}{p}$ *determines* the magnitude of the interval PQ; for it follows from the proposition of Art. **21** that no interval can be equal to PQ unless the numbers of vibrations of its tones have the same ratio.

But although this ratio *determines* the interval, it cannot be taken as a *measure* of the interval, as we shall now shew.

Suppose Q is higher than P, and let R be a third tone higher than Q, having r vibrations in a second. Then the interval PR is the sum of the intervals PQ, QR; and therefore any number taken as a measure of PR ought to be the sum of the numbers taken by the same rule as the measures of PQ, QR. Now the ratios $\frac{q}{p}$, $\frac{r}{q}$, $\frac{r}{p}$, which *determine* the three intervals PQ, QR, PR, do not fulfil this condition, for $\frac{q}{p}+\frac{r}{q}$ is not equal to $\frac{r}{p}$.

But if we take for the measure of an interval, not the ratio of the numbers of vibrations, but the *logarithm* of that ratio, the required condition is satisfied; for $\frac{r}{p} = \frac{q}{p} \times \frac{r}{q}$, and therefore the logarithm of $\frac{r}{p}$ is the sum of the logarithms of $\frac{q}{p}$ and $\frac{r}{q}$.

23. If then p, q be the numbers of vibrations, in a given time, of two tones P, Q, the logarithm of $\frac{q}{p}$ may properly be taken as a numerical measure of the interval PQ. And in order to compare the magnitudes of different intervals we must compare, not the corresponding ratios, but the logarithms of those ratios. The base of the system of logarithms may be any whatever. In fact the choice of the base merely determines what interval shall be represented numerically by unity, since the logarithm of the base is 1 in every system. Thus, if we took 2 for the base, we should have log 2 = 1; and therefore 1 would be the measure of the interval which has 2 (or $\frac{2}{1}$) for the ratio of the vibrations of its tones. In other words, the octave (see Art. **31**) would then be the *unit-interval.* There would be some advantage in this; but practically it is convenient to use the common logarithms, of which the base is 10.

24. Though the ratio $\frac{q}{p}$ cannot be used as a measure, it is often convenient to use it as the *name* of the interval PQ. When it is so used we shall inclose the fraction in brackets thus, 'the interval $\left(\frac{q}{p}\right)$;' the brackets being intended to remind the reader that the measure of the interval is not $\frac{q}{p}$, but $\log \frac{q}{p}$.

25. The interval between two tones P, Q, may, like the interval between two points in a line, be reckoned in two opposite ways, namely, from P to Q, or from Q to P; and this difference will henceforth be indicated by calling the same interval PQ or QP accordingly.

And if we introduce the signs + and − in the same way as in modern elementary Geometry, it is evident that the rules for the addition and subtraction of straight lines (in the same direction) may be applied at once to the addition and subtraction of intervals. Thus, $QP = -PQ$, or $PQ + QP = O$. And if P, Q, R be any three tones whatever, then

$$PR = PQ + QR = QR - QP;$$

and

$$PQ + QR + RP = O, \ \text{\&c.}$$

26. In designating any interval by the corresponding ratio as a name, we shall put that number in the denominator which is proportional to the number of vibrations producing the tone *from* which the interval is reckoned.

Thus, 'the interval $\left(\frac{q}{p}\right)$' will mean PQ, while 'the interval $\left(\frac{p}{q}\right)$' will mean QP.

Then also $\log \frac{q}{p}$ being taken as the measure of PQ, $\log \frac{p}{q}$ will be the measure of QP with its proper sign; for $\log \frac{p}{q} = -\log \frac{q}{p}$.

The equations $\frac{r}{p} = \frac{q}{p} \times \frac{r}{q}$, $\frac{r}{p} = \frac{r}{q} \div \frac{p}{q}$, compared with $PR = PQ + QR$, $PR = QR - QP$, shew that addition and subtraction of intervals correspond to multiplication and division of ratios; the words addition and subtraction being no longer restricted to their arithmetical sense, but used in the same way as in the geometry of a straight line.

In fact this is only another way of saying that the logarithm of the ratio is a proper measure (both as to magnitude and sign) of the interval.

It follows, from the conventions above made, that those intervals are to be considered *positive* which are reckoned from a lower to a higher tone; since the logarithm of a ratio greater than unity is positive.

27. It will be a convenient abridgment to call the ratio which determines an interval the 'ratio of the interval,' though the expression is in itself unmeaning.

28. The interval between the lowest and highest audible tones is theoretically capable of unlimited subdivision by the interposition of intermediate tones, though there is a limit to the power of the ear to distinguish nearly coincident tones, as there is to the power of the eye to distinguish nearly coincident tints.

A series of tones at finite intervals, selected according to some definite law, is usually and appropriately called a *scale;* for the selected tones are the steps of a ladder by which we ascend from a lower to a higher pitch.

A scale formed by taking an unlimited succession of tones, produced by vibrations of which the numbers (in a given time) are proportional to 1, 2, 3, 4, &c. is called the scale of *Natural Harmonics*. We shall henceforward refer to it simply as 'the harmonic scale.'

The different notes which can be produced from a simple tube, used as a trumpet, belong to a scale of this kind. And each note of the trumpet, of the human voice, of a vibrating string, in short every musical note produced in any of the most usual ways, is compounded of simple tones also belonging to a harmonic scale. The explanation of these facts on mechanical principles will be given afterwards; at present we state them as a reason for giving here a general view of this primary and fundamental scale.

29. The absolute pitch of the lowest or fundamental tone may be any whatever. If we choose the thirty-third part of a second for the period of this tone, it will have the same pitch as the lowest C of a modern pianoforte, according to a standard now very commonly adopted[1]. (If we chose a different period we should of course merely transpose the scale without altering the intervals.)

The series of tones will then begin as follows:

The numbers written above the notes are the numbers of vibrations in the thirty-third part of a second.

[1] This is the German pitch. In England there is at present no uniform standard.

Those notes which are marked with an asterisk do not exactly represent the corresponding tones, but are the nearest representatives which the modern notation supplies.

30. The letters below the notes are, with a slight alteration, those used by German writers. In this system of notation, C, D, E, F, G, A, B, represent the seven notes beginning with the second C (reckoning upwards) of a modern pianoforte, or the lowest note of a violoncello.

The octaves above these notes are represented by the small letters c, d, e, f, g, a, b. Thus, c is the lowest note of a viola, and the lowest c of a tenor voice; and g is the lowest note of a violin.

Higher notes than b are represented by putting accents above the small letters; and lower notes than C by putting accents below the capitals. Each accent above a small letter raises the tone by an octave; and each accent below a capital lowers the tone by an octave.

Thus, g'' is the highest g of an ordinary soprano voice, $C_{,}$ is the lowest C of a modern pianoforte, and $C_{,,}$ is the so-called 32-foot C of an organ. The compass of the newest pianofortes is from $A_{,,}$ to a'''', &c.

As we shall use this notation for the future, the reader is recommended to become familiar with it.

(German writers use h instead of our b, and b instead of our ♭b.)

31. The intervals between the several tones of the harmonic scale have in a few cases received names derived from the places of the tones in the diatonic scale (Art. **35**).

The most important of these intervals, with their ratios, are:

Octave	$C_{,}$	—	C	$\frac{2}{1}$
Fifth	C	—	G	$\frac{3}{2}$
Fourth	G	—	c	$\frac{4}{3}$
Major third	c	—	e	$\frac{5}{4}$
Minor third	e	—	g	$\frac{6}{5}$
	g	—	♭b*	$\frac{7}{6}$
	♭b*	—	c'	$\frac{8}{7}$

Major second	c′	—	d′	. .	$\frac{9}{8}$
Minor second	d′	—	e′	. .	$\frac{10}{9}$
Diatonic semitone	b′	—	c″	. .	$\frac{16}{15}$

The two intervals ($\frac{7}{6}$), ($\frac{8}{7}$) are not formally recognized in modern music, though they are probably often used both by singers and by players on instruments not restricted to a finite number of notes.

The major and minor second are often called major and minor *tones;* but it is desirable to avoid the use of the word tone in two senses. Other intervals (such as the sixth, &c.), formed by addition and subtraction of those in the above list, may be omitted for the present.

32. The second and all following tones of the harmonic scale are often called the 'harmonics' of the first or fundamental tone. The second tone is the *first harmonic*, the third tone is the *second harmonic*, and so on.

If we take any tone of the harmonic scale as a new fundamental tone, the whole series of its harmonics will be found in the original scale.

Thus the series 3, 6, 9, 12 . . . ., gives the harmonic scale of the third tone; and in general the series

$$n,\ 2n,\ 3n,\ 4n,\ \ldots\ldots,$$

gives the harmonic scale of the nth tone.

33. Omitting for the present any discussion of the character of other intervals, we may here notice a peculiar property which distinguishes the first interval in the harmonic scale, the *octave*. When we compare two tones which differ by an octave, we are affected by a certain sense of *sameness*, which we do not feel in the case of any other interval unless it be a multiple of an octave. A supposed explanation of this property will be discussed hereafter; but, whether it be explicable or not, it entitles the octave to be regarded as a natural unit (see Art. **23**) with which other intervals may be compared. At the same time it gives a periodic character to the scale: every tone which has occurred once seems to occur again and again at equal intervals.

34. If we compare the intervals into which any two successive octaves of the harmonic scale are divided, we see that every interval between consecutive tones in one octave is divided into two intervals in the next higher octave.

The law of this subdivision is worth observing. The ratio of the interval between consecutive tones is always of the form $\frac{n+1}{n}$; now $\frac{n+1}{n} = \frac{2n+2}{2n} = \frac{2n+1}{2n} \times \frac{2n+2}{2n+1}$; in other words (see Art. **26**) the interval $\left(\frac{n+1}{n}\right)$ is the sum of the two intervals $\left(\frac{2n+1}{2n}\right)$, $\left(\frac{2n+2}{2n+1}\right)$, and it is in fact divided into these two in the next higher octave; for in that octave occur three consecutive tones corresponding to the numbers $2n$, $2n+1$, $2n+2$, of vibrations.

Thus, the first octave, C, — C, is undivided. The second octave, C — c, is divided into a fifth and a fourth,

$$\tfrac{2}{1} = \tfrac{4}{2} = \tfrac{3}{2} \times \tfrac{4}{3}.$$

In the third octave the fifth, c — g, is divided into a major and a minor third, $\frac{3}{2} = \frac{6}{4} = \frac{5}{4} \times \frac{6}{5}$;
whilst the fourth, g — c′, is divided into two intervals, $(\frac{7}{6})$, $(\frac{8}{7})$, which have not received names. In the fourth octave the major third, c′ — e′, is divided into a major and a minor second,

$$\tfrac{5}{4} = \tfrac{10}{8} = \tfrac{9}{8} \times \tfrac{10}{9}.$$

Thus every interval is divided into a major and minor half (if the word half may be so used), according to a law which may be called the law of *natural bisection*, the major half being always the lower.

35. The theory of artificial scales cannot be discussed here; but it will be useful to state, without reference to theory, the actual construction of the modern diatonic major scale. If we take two tones at an interval of a fifth, and the intermediate tone which bisects the fifth naturally (Art. **34**), for example, c, e, g, we obtain three tones which when sounded together produce a *triad* or common chord. And if we take three such triads one above another, so that the highest tone of the first is the lowest of the second, and the highest of the second the lowest of the third, we obtain seven tones, rising one above another by alternate major and minor thirds, thus:

$$\begin{array}{ccccccccccccc} F & - & A & - & c & - & e & - & g & - & b & - & d' \\ & \frac{5}{4} & & \frac{6}{5} & & \frac{5}{4} & & \frac{6}{5} & & \frac{5}{4} & & \frac{6}{5} & \end{array}$$

Lastly, if we take the lowest tone (c) of the middle triad as the so-called *tonic* or first tone of the scale, and bring all the other tones within the compass of an octave by substituting f, a, d, for F, A, d′, we obtain the seven tones of the diatonic scale, to which an eighth (viz. c′, the octave of the tonic) is usually added, thus: c, d, e, f, g, a, b, c′.

Returning, however, to the above system of triads, let us find the ratios of the intervals from the lowest tone, F, to each of the other tones. This will be done by successive multiplication of the ratios $\frac{5}{4}$, $\frac{6}{5}$, &c. (Art. **26**), and thus we obtain the following ratios for the intervals:

$$\begin{array}{ccccccc} F, & A, & c, & e, & g, & b, & d'. \\ \frac{1}{1}, & \frac{5}{4}, & \frac{3}{2}, & \frac{15}{8}, & \frac{9}{4}, & \frac{45}{16}, & \frac{27}{8}. \end{array}$$

Here the ratio written under each tone is that of the interval from F to that tone.

If we reduce all these fractions to the least possible common denominator, viz. 16, the numerators will be the smallest whole numbers proportional to the numbers of vibrations of the corresponding tones, thus:

F,	A,	c,	e,	g,	b,	d′.
16,	20,	24,	30,	36,	45,	54.

Recollecting now that the number of vibrations of f is double that of F, &c., we obtain the following series for the diatonic scale:

c,	d,	e,	f,	g,	a,	b,	c′.
24,	27,	30,	32,	36,	40,	45,	48.

36. The tones of the diatonic scale have all received technical names, of which it is sufficient to mention three, viz. the *tonic* or first tone of the scale, the *dominant* or fifth tone, and the *subdominant* or fourth tone. Thus, in the above scale c is the tonic, g the dominant, and f the subdominant.

37. The tones of a diatonic scale, having for their actual numbers of vibrations those given at the end of Art. **35**, belong evidently to the harmonic scale (Art. **28**), of which the fundamental tone has *one* vibration. This fundamental tone is five octaves below the subdominant, for $\frac{32}{1} = (\frac{2}{1})^5$.

Hence, neglecting the difference between tones which differ by a whole number of octaves, we may say that *the diatonic scale is selected from the harmonic scale of its subdominant.*

This proposition is to be understood merely as the statement of a mathematical fact, and not as involving any theory of the actual derivation of the scale.

The diatonic scale can only be represented in whole numbers when the number of vibrations of its tonic is divisible both by 3 and 8.

Hence we may say that any harmonic scale contains the tones (or their octaves) of the diatonic scales of all those amongst its tones which correspond to multiples of 3, and of no others.

The series of multiples of 3, viz. 3, 6, 9, 12, ..., gives a harmonic scale which has for its fundamental tone the third tone of the original scale.

Thus, every harmonic scale may be said to contain the diatonic scales of its third tone, and of all the harmonics of that tone.

38. Returning now to the series of whole numbers which represent the diatonic scale (Art. **34**), we find for the intervals between successive tones of the scale the following ratios:

c . . . . d,	$\frac{27}{24} = \frac{9}{8}$	(major second).
d . . . . e,	$\frac{30}{27} = \frac{10}{9}$	(minor second).
e . . . . f,	$\frac{32}{30} = \frac{16}{15}$	(diatonic semitone).
f . . . . g,	$\frac{36}{32} = \frac{9}{8}$	(major second).
g . . . . a,	$\frac{40}{36} = \frac{10}{9}$	(minor second).
a . . . . b,	$\frac{45}{40} = \frac{9}{8}$	(major second).
b . . . . c′,	$\frac{48}{45} = \frac{16}{15}$	(diatonic semitone).

39. In this scale the octave is divided into two so-called tetrachords, c, d, e, f; g, a, b, c′, separated by the major second f . . . . g. These tetrachords are nearly, but not exactly, alike; for the intervals in the lower and upper tetrachords are:

(lower) major second, minor second, semitone;
(upper) minor second, major second, semitone.

Hence, the upper tetrachord of the scale of c is not exactly identical with the lower tetrachord of the scale of g; for the a in the latter scale is a major second above g, and in the former a minor second only. The a of the scale of c is therefore flatter than that of the scale of g by the difference between a major and a minor second. This difference is called a *comma;* and the ratio of a comma is (Art. **26**)

$$\tfrac{9}{8} \div \tfrac{10}{9} = \tfrac{81}{80}.$$

A comma is nearly equal to the fifth part of a diatonic semitone; for $(\frac{81}{80})^5$ is nearly equal to $\frac{16}{15}$.

These details belong more properly to another chapter; but they have been given here in order to shew at once, by a simple example, the imperfection of the ordinary musical notation for all but practical purposes. The letter *a* and the note  are made to serve for the scales both of c and g, to say nothing of other scales; whereas in fact the letters, as ordinarily used, and the notes on the stave, are capable of representing accurately one diatonic scale and no more.

The whole structure, however, of modern music is founded on the possibility of educating the ear not merely to tolerate or ignore, but even in some degree to take pleasure in, slight deviations from the perfection of the diatonic scale.

40. The relation between intervals and numerical ratios may be illustrated by the curve called the logarithmic (or equiangular) spiral. This curve (see Fig. 5) consists of convolutions, of which the number reckoned from any point, either inwards or outwards, is infinite. Inwards, the curve approaches continually nearer to a point O (called the pole), which it never actually reaches. Outwards, it recedes from the pole without limit. These two properties are common to many spirals. But the curve in question has this particular property, that if straight lines be drawn from the pole to any two points on the curve, the logarithm of the ratio of their lengths is proportional to the angle between them. It follows (Art. **23**), that if the lengths of any two such lines, as OA, OB, be taken to represent numbers of vibrations in a given time, the angle AOB may be taken as a measure of the interval between the tones produced by those vibrations.

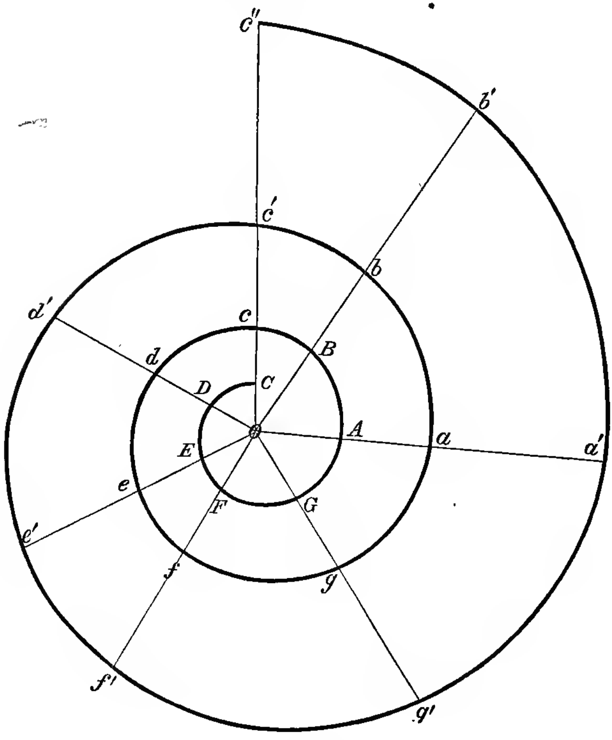

Fig. 5.

The curve can be so drawn, and is so drawn in the figure, that a complete revolution doubles the distance from the pole. Thus, Oa is double of OA, and Oa' of Oa, &c.; so that a point, starting from A and following the curve outwards, doubles its distance from O whenever it crosses the line OA produced, that is, whenever it completes a revolution. But, when the number of vibrations is doubled, the tone is raised an octave. Hence the angle described in a complete revolution, that is, four right angles or 360°, represents an octave; and any fraction or multiple of four right angles represents the same fraction or multiple of an octave.

We may, therefore, consider a point, following the curve outwards, to represent a tone continually rising in pitch; and successive passages of the point through any given line drawn from the pole will represent passages of the tone through successive octaves. Thus the geometrical periodicity of the curve presents to the eye a sort of picture of the periodicity perceived by the ear in a continuously rising tone.

The angles representing the intervals used in the diatonic scale cannot (with the exception of the octave) be exactly expressed in degrees, minutes, and seconds; but, to the nearest second, they are as follows[1]:

	°	″	′
Octave	360	0	0
Fifth	210	35	11
Fourth	149	24	49
Major third	115	53	38
Minor third	94	41	33
Major second	61	10	22
Minor second	54	43	16
Diatonic semitone	33	31	11
(Comma)	6	27	6

[1] The equation to the curve is $r = a \cdot 2^{\frac{\theta}{2\pi}}$; hence, if r, r', be two radii nectores in the ratio of $m : n$, we have

$$\frac{m}{n} = 2^{\frac{\theta - \theta'}{2\pi}};$$

hence
$$\theta - \theta' = 2\pi \frac{\log m - \log n}{\log 2};$$

which gives the angular measure of the corresponding interval. Thus, for the fifth, $\frac{m}{n} = \frac{3}{2}$; hence the angle (in degrees) is

$$360° \times \frac{\log 3 - \log 2}{\log 2}.$$

The angles being known for the fifth and the fourth, the rest of those which occur in the diatonic scale can he found by addition and subtraction.

41. In Fig. 5 the letters represent the tones of three octaves of the diatonic scale. It may be observed that the same figure might be used to represent a descending instead of an ascending scale. The distances from the pole would then be proportional to the *periods* of the tones, or (as will be shewn hereafter) to the lengths of the strings, of given kind and given tension, which produce them. The angles would then have to be taken in the reverse order.

42. A more usual mode of representing intervals graphically is by parts of a straight line, of which the whole is assumed to represent an octave. When this method is adopted it is convenient to divide the octave line into equal parts, of which the number is approximately the product of some power of 10 by the logarithm of 2. Thus, if we divide it into 301 parts (301 being $10^3 \times \log 2$ nearly) the logarithm of the ratio defining the interval measured by n divisions will be, approximately, $\frac{n}{1000}$. Conversely, the number of divisions representing any interval will be 1000, multiplied by the logarithm of the ratio of that interval. Thus, a comma will be represented by

$$1000 \times \log \tfrac{81}{80} = 5 \text{ divisions nearly.}$$

A division on this scale corresponds to something more than a degree on the scale described in Art. **40.** Dr. Young divided the octave into 30,103 parts, and Mr. De Morgan has proposed to call each of these parts an 'atom.' An atom would be very nearly equal to 43″ on the scale of Art. **40.** (See De Morgan, 'On the Beats of Imperfect Consonances,' *Camb. Phil. Trans.* vol. x. p. 4.)

CHAPTER III.

COMPOSITION OF VIBRATIONS.

43. THE simplest type of periodic motion is afforded by a point describing a circle with a constant velocity. For this is the only kind of motion in which the velocity and the change of direction are both uniform.

Such a motion may be considered as the simplest possible vibration of a point (Art. **17**). We shall call it, for the present, a simple circular vibration.

44. The vibration of a point, in its most general form, may be defined as motion in a curve which returns into itself, with a velocity which is always the same at the same point of the curve.

Suppose the curve to be plane, such, for instance, as in Fig. 1. Draw arbitrarily any two axes, OX, OY. (It is immaterial whether they meet the curve or not). From P, the position of the moving point at any time, draw PM, PN, parallel to OY, OX. As P moves round and round the curve, the point M will move backwards and forwards in the line OX with a periodic rectilinear motion; that is, M will perform rectilinear vibrations. In the same way N will perform rectilinear vibrations in OY.

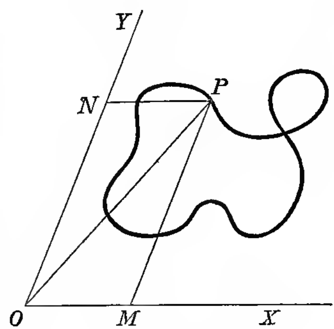

Fig. 1.

The curvilinear vibration of P is said to be *compounded* of the rectilinear vibrations of M and N.

In fact, if we consider P as a point *displaced from* O, the displacement OP is compounded of the two displacements OM, ON, in the sense explained in Art. **9.** It is evident that any given vibration of a point in a plane may in this manner be resolved into two component rectilinear vibrations, in an infinite variety of ways; and the character of each component depends in general upon the directions of both the axes.

When nothing is said to the contrary, it is assumed that the axes are rectangular. In this case PM is perpendicular to OX; and the vibration of M (the orthogonal projection of P) is then called absolutely *the rectilinear component*, in the direction OX, of the vibration of P.

45. Any vibration of a point, whether plane or not, may be similarly resolved into *three* component rectilinear vibrations by taking three arbitrary axes, OX, OY, OZ. The point M, in OX, is then determined by drawing PM parallel to the plane YOZ. The projections of P on OY, OZ, are to be found in the same way, *mutatis mutandis*.

46. We will now return to the case of a simple circular vibration (Art. **43**). Such a vibration is completely determined when four things are given:

(1) The *period*, or time of describing the whole circle.
(2) The radius of the circle.
(3) The position of the moving point at some one given instant.
(4) The *direction of motion* (whether right-handed, or left-handed).

The *phase* of the vibration is defined by the angle between an arbitrary fixed radius and the radius of the moving point at any time.

Hence, the third of the above data may be expressed by saying that the phase is given at a given instant. The fourth datum will seldom have to be referred to.

Through the centre of the circle (Fig. 2) draw any two rectangular axes, OX, OY, and from P, the position of the moving point at any time, draw PM, PN, perpendicular to the axes. (The origin is taken at the centre for convenience, but it might be anywhere.) Then, as P describes the circle with a uniform motion, M vibrates in the line AA', and N in BB', and it is evident that these two rectilinear vibrations are perfectly similar.

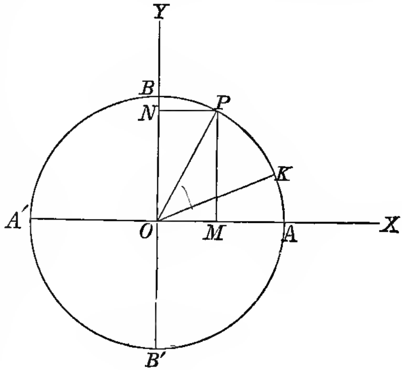

Fig. 2.

Vibrations of the kind performed by M or N, that is, rectilinear components of simple circular vibrations, are distinguished by many remarkable properties, and may properly be considered as the simplest kind of rectilinear vibrations. This will appear more clearly as we proceed.

47. Let an arbitrary radius, OK, be taken as a fixed direction from which to measure the angle θ (or KOP) described by OP, and suppose the direction of the motion to be such that θ increases with the time. Then, putting a for the radius of the circle, and α for the angle KOA, we have

$$ON = a \sin (\theta + \alpha).$$

Suppose the time, t, is reckoned from the instant of a passage of P through K, and let τ be the period of the vibration, or time of describing the whole circumference. Then the time of describing the arc KP is $\frac{\theta}{2\pi}\tau$, and when the moving point is at P the value of t must be $t = \frac{\theta}{2\pi}\tau + n\tau$, n being some whole number; whence we have $\theta = \frac{2\pi t}{\tau} - 2n\pi$. Introducing this value of θ in the above value of ON, and putting $ON = y$, we find

$$y = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right) \tag{1}$$

as the equation which determines the position of N at the time t.

In like manner, putting $OM = x$, we should find

$$x = a \cos\left(\frac{2\pi t}{\tau} + \alpha\right); \tag{2}$$

and, if the latter equation be put in the form

$$x = a \sin\left(\frac{2\pi t}{\tau} + \frac{\pi}{2} + \alpha\right),$$

it is seen that both (1) and (2) represent vibrations of the same kind, but that x and y take the same values at *different times*. In fact, the value of x at time t is the same as that of y at time $t + \frac{\tau}{4}$, so that the first vibration may be said to be a quarter of a period behind the second.

48. Vibrations of the kind considered in the last article, namely, rectilinear vibrations in which the displacement of the moving point at the time t can be represented by an expression of the form

$$a \sin\left(\frac{2\pi t}{\tau} + \alpha\right), \tag{3}$$

may be conveniently called *rectilinear harmonic vibrations.*

The constant a is called the *amplitude*, because its value is that of the greatest displacement.

The angle $\frac{2\pi t}{\tau} + \alpha$ is called the *phase* of the vibration. The constant α is therefore known if the phase at any given time be given.

49. A rectilinear harmonic vibration can be resolved into two others of the same kind in an infinite variety of ways. Usually the directions of the two components are taken at right angles to one another. In this case the component of (3)

(Art. **48**), in a direction making the angle ω with the direction of (3), is evidently

$$a \cos \omega \, . \, \sin\left(\frac{2\pi t}{\tau} + \alpha\right).$$

Composition of Vibrations.

50. Rectilinear harmonic vibrations may be compounded according to the general law of the superposition of displacements explained in Art. **9**. The two cases of most importance are those in which the two vibrations to be compounded are, (1) in the same direction, (2) in directions at right angles to one another.

First, then, let the vibrations be in the same direction; in this case the resultant displacement is simply the algebraical sum of the component displacements. If the component vibrations be represented by the expressions

$$a \sin\left(\frac{2\pi t}{\tau} + \alpha\right), \qquad b \sin\left(\frac{2\pi t}{\tau'} + \beta\right),$$

the resultant displacement will be represented by the sum

$$a \sin\left(\frac{2\pi t}{\tau} + \alpha\right) + b \sin\left(\frac{2\pi t}{\tau'} + \beta\right).$$

This expression is *periodic* if the periods τ and τ' be commensurable; for its value will then evidently be unaltered if t be increased by any common multiple of τ and τ'. But it does not in general admit of any useful reduction.

If, however, the periods τ, τ', of the component vibrations be *equal*, the resultant motion is itself a harmonic vibration having the same period. For in this case the above expression may be written in the form

$$(a \cos \alpha + b \cos \beta) \sin \frac{2\pi t}{\tau} + (a \sin \alpha + b \sin \beta) \cos \frac{2\pi t}{\tau};$$

and this is identical with

$$A \sin\left(\frac{2\pi t}{\tau} + B\right),$$

provided A and B be so taken that

$$A \cos B = a \cos \alpha + b \cos \beta,$$
$$A \sin B = a \sin \alpha + b \sin \beta,$$

which equations are satisfied by

$$A = \sqrt{a^2 + b^2 + 2\,a\,b\cos(\alpha - \beta)},$$

$$B = \tan^{-1}\frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta},$$

where we may suppose the square root to be taken *positively.*

The value of A, the amplitude of the resultant vibration, depends both on the amplitudes a, b, of the component vibrations, and also on the angle $\alpha \backsim \beta$, which is the *difference of their phases.*

If $\alpha - \beta = 0$, the component vibrations are always in the same phase, and $A = a + b$; or the amplitude of the resultant is the sum of the amplitudes of the components.

If $\alpha \backsim \beta = \pi$, one of the component vibrations is half a period behind the other, and $A = a \backsim b$; or the amplitude of the resultant is the difference of amplitudes of the components. In this latter case, if $a = b$, then $A = 0$, and the component vibrations completely destroy one another.

51. Next, let the component vibrations be at right angles to one another, and let their periods be τ, τ'; putting for convenience $\frac{2\pi}{\tau} = n$, $\frac{2\pi}{\tau'} = n'$, we may represent them by the equations

$$x = a\sin(nt + \alpha), \qquad y = b\sin(n't + \beta),$$

and these are the co-ordinates, at the time t, of the moving point (see Art. **48**).

The equation to the locus of the point would be found by eliminating t between these two equations. The elimination can, theoretically, be performed so as to lead to an algebraical equation between x and y, whenever τ and τ' are commensurable, though the process is impracticable except in simple cases.

The simplest case of all is that in which the periods of the component vibrations are equal. We have, then, to eliminate t between the two equations

$$x = a\sin(nt + \alpha), \qquad y = b\sin(nt + \beta); \qquad (4)$$

from them we have

$$\cos \alpha \sin nt + \sin \alpha \cos nt = \frac{x}{a},$$

$$\cos \beta \sin nt + \sin \beta \cos nt = \frac{y}{b};$$

and if the values of $\sin nt$, $\cos nt$, be found from these equations, and the sum of their squares be equated to 1, the result is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{xy}{ab} \cos(\alpha - \beta) - \sin^2(\alpha - \beta) = 0. \quad (5)$$

52. This equation always represents an ellipse, which may however degenerate into a circle (when the axes are equal), or a straight line (when the length of one axis is 0).

The dimensions and position of the ellipse depend only on the amplitudes (a, b) and the difference of phases $(\alpha \backsim \beta)$ of the component vibrations. If the component vibrations are always in the same phase, that is, if $\alpha - \beta = 0$, the equation becomes

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0,$$

and the ellipse degenerates into a straight line, or, more strictly, into two coincident straight lines. Again, if $\alpha \backsim \beta = \pi$, that is, if one of the component vibrations be *half* a period behind the other, the equation becomes

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0,$$

and the locus is again a straight line.

If $\alpha \backsim \beta = \frac{\pi}{2}$, or $\frac{3\pi}{2}$, that is, if one component be a *quarter* of a period behind the other, the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the axes of the ellipse coincide with the directions of the component vibrations.

53. It is evident from equations (4) that the ellipse described by the vibrating point is always inscribed in a rectangle, of which the sides are $2a$, $2b$, since the extreme values of x and

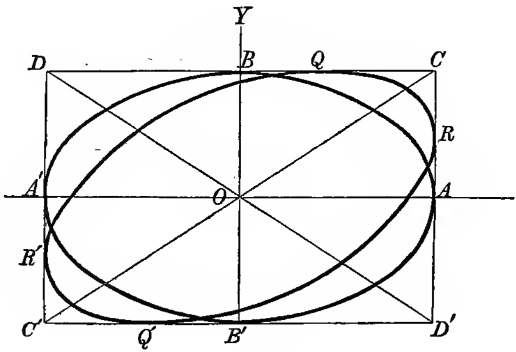

Fig. 3.

y are $\pm a$, $\pm b$ (see Fig. 3). This may be formally proved from equation (5); for if in that equation we put $y = b$, the result is

$$\left(\frac{x}{a} - \cos(\alpha - \beta)\right)^2 = 0,$$

shewing that the ellipse meets the line $y = b$ in two coincident points (as at Q), of which the abscissa is $a \cos(\alpha - \beta)$. Similarly, by putting $x = a$, we find $b \cos(\alpha - \beta)$ for the ordinate of the point of contact R.

When $\alpha = \beta$, Q and R coincide at C, and the ellipse degenerates into the diagonal CC'; and when $\alpha \sim \beta = \pi$, it degenerates in like manner into the diagonal DD'.

An ellipse inscribed in a given rectangle is completely determined if one point of contact be given. Hence, it follows that if the angle $\alpha - \beta$ varied continuously from 0 to π, the ellipse would pass continuously through every form capable of being inscribed in the rectangle $DCD'C'$, including the two diagonals as extreme cases.

54. Let us now suppose BB' (Fig. 3) to be the axis of a cylinder, of which DC, $C'D'$ are the circular ends seen edgewise by an eye placed at a distance in a line through O perpendicular to the plane of the paper. Suppose also the cylinder to be transparent, and to have marked on its surface the trace of a section made by a plane touching the circular edge of each end. This section would be an ellipse, which, as seen by the eye, would be orthogonally projected into another ellipse inscribed in the rectangle $D\,C\,D'\,C'$. And if the cylinder were turned uniformly about its axis, this inscribed ellipse would evidently go through all its possible forms in such a manner that the distance $B\,Q$ would be equal to a multiplied by the cosine of a uniformly varying angle.

Comparing these results with those of the last article, we see that this construction gives a perfect representation of the change undergone by the ellipse which is the resultant of two rectilinear harmonic vibrations at right angles to one another, when the difference of phase of the component vibrations varies uniformly; the angular velocity of the rotating cylinder being equal to the rate of variation of the difference of phase. (For $B\,Q = a \cos(\alpha - \beta)$).

55. A similar construction may be employed in the much more general case of any two rectilinear vibrations of which the periods are commensurable, and of which one, at least, is harmonic; a case which could rarely be treated algebraically (Art. **50**).

For the sake of clearness, let us suppose the direction of the harmonic vibration to be horizontal, or in the axis of x. This vibration will therefore be represented (Art. **47**) by an equation of the form

$$x = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right). \tag{6}$$

Let f denote any periodic function, which is always finite, and such that $f(z) = f(z + 2\pi)$. Then the vertical vibration may be represented by an equation of the form

$$y = f\left(\frac{2\pi t}{\tau'} + \beta\right). \tag{7}$$

Let us also suppose the ratio of the periods τ, τ' to be such that $n\tau' = m\tau$, m and n being integers.

Now let a curve be drawn in which the abscissa of any point is proportional to the time t, and the ordinate is the corresponding value of y given by equation (7). (Fig. 4, in which $AB = \tau'$, may be taken to represent two periods of such a curve.) And suppose the unit-line to have been so chosen

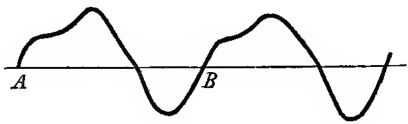

Fig. 4.

that a line equal to $n \times AB$ shall be equal to m times the circumference of a circle of which the radius is a. If, then, a rectangular slip of paper were cut out, containing exactly n periods of the curve, it could be rolled m times round a cylinder of radius a. We should thus obtain a complicated curve on the surface of the cylinder. Suppose the whole to become transparent, and to be viewed in the manner explained in Art. **53**; then if a point P were to describe the curve in such a manner that the projection of P on the base of the cylinder should describe the circumference of the circle uniformly, the horizontal motion of P, as seen by the eye, would be a harmonic vibration, while its vertical motion would be identical with that defined by equation (7). Moreover, m horizontal vibrations would be completed in the same time as n vertical vibrations.

Hence the apparent co-ordinates of P would be always identical with the values of x and y in equations (6) and (7), provided the cylinder were turned (if necessary) about its axis into such a position as to give x and y corresponding values at any one instant; and thus we should get, as before, a perfect representation of the resultant vibration.

56. An alteration in the value of either of the angles α or β (equations (6), (7)) would alter the time at which the corresponding vibration passes through a given phase, and cause different values of x and y to become contemporaneous. And the same effect would be produced to the eye by turning the cylinder about its axis into a new position.

Thus, if α were changed into $\alpha + \epsilon$, the effect would be the same as if the cylinder were turned backwards (that is, with a motion contrary to that of the projection of P) through an angle ϵ, since in either case the passage of the horizontal vibration through a given phase would be accelerated by a time $\frac{\epsilon}{2\pi}\tau$. A retardation of the vertical vibration would have the same effect as an equal acceleration of the horizontal vibration; now, if β were changed into $\beta - \epsilon'$, the retardation would be $\frac{\epsilon'}{2\pi}\tau'$, and this would be equal to the former acceleration, if $\epsilon'\tau' = \epsilon\tau$, or $\epsilon' = \frac{\tau}{\tau'}\epsilon$.

Hence, a change of β into $\beta + \epsilon'$ is equivalent to a change of α into $\alpha + \frac{\tau'}{\tau}\epsilon'$, that is, to a turning of the cylinder backwards through an angle $\frac{\tau'}{\tau}\epsilon'$, or $\frac{m}{n}\epsilon'$.

57. Since the curve rolled on the cylinder consists of n similar portions, the angular distance between corresponding points on two consecutive portions is $\frac{2\pi}{n}$.

Hence, if the cylinder were turned continuously, the projected curve seen by the eye would go continuously through all its possible forms during a rotation of the cylinder through the angle $\frac{2\pi}{n}$.

We see also that a uniform variation either of α, or of β, would have the same effect as a uniform rotation of the cylinder[1].

58. The results obtained in Arts. **52–55** may be exhibited to the eye. But before explaining the mode of doing so, we

[1] This mode of representing stereoscopically the composition of vibrations is due to M. Lissajous. See his memoir 'Sur l'Étude optique des Mouvements vibratoires,' *Ann. de Ch. et de Phys.*, t. 51, p. 147.

will consider the case in which the ratio of the periods of the two vibrations is *nearly*, but not exactly, expressed by a given numerical fraction.

Suppose then, as before, that τ, τ' are given constant quantities, such that $m\tau = n\tau'$ (the ratio $m : n$ being in its lowest terms). And suppose that the vertical vibration is the same as before (see equation (7), Art. **54**), but that the horizontal (harmonic) vibration is now defined by the equation

$$x = a \sin\left(\frac{2\pi}{\tau}(1 + k)t + \alpha\right),$$

so that its period is no longer τ, but $\frac{\tau}{1+k}$, which will differ slightly from τ, if k be a small quantity, positive or negative.

If now we put $\alpha + \frac{2\pi k}{\tau} t = \alpha'$, the above equation becomes

$$x = a \sin\left(\frac{2\pi t}{\tau} + \alpha'\right),$$

and the result of eliminating t between this last equation and (7) would be the same as before, except that α would be changed into α', that is, into a uniformly varying instead of a constant quantity. We may, therefore, consider that the vibrating point still describes the curve (8), but that the curve is *disturbed* by the variation of α, just as we consider a planet to describe an ellipse disturbed by the variation of one or more of its elements.

59. It has been seen (Art. **57**) that a uniform variation of α would have the same effect as a uniform rotation of the cylinder with an angular velocity equal to the rate of variation of α, that is, in the case now supposed, to $\frac{2\pi k}{\tau}$; and also that the curve, as seen by the eye, would go through all its forms while the cylinder turned through an angle $\frac{2\pi}{n}$. Hence the curve would go through all its forms in a time equal to $\frac{2\pi}{n} \div \frac{2\pi k}{\tau}$, or $\frac{\tau}{nk}$. Suppose that M, N, are the actual numbers, in a unit of time, of the horizontal and vertical vibrations; then $M = \frac{1+k}{\tau}$, and $N = \frac{1}{\tau'} = \frac{n}{m\tau}$.

Now the number of complete cycles of change of the curve in the unit of time is $\frac{nk}{\tau}$, and this is equal to

$$nM - mN. \qquad (9)$$

This expression is worthy of remark, especially in connection with the theory of the so-called *beats* of imperfect consonances, as will appear afterwards.

The *sign* of $nM - mN$, being the same as that of k, is positive or negative according as the ratio $M : N$ is greater or less than $m : n$. In the former case, the rotation of the cylinder is backwards; in the latter, forwards.

In the particular case in which the periods of the vibrations are nearly *equal*, or $m = n = 1$, the expression (9) becomes simply $M - N$. Or the number of cycles of change in a unit of time is the *difference of the actual numbers of vibrations.*

60. The two Figures, 5, 6, illustrate the case in which both vibrations are harmonic and have the same amplitude, while

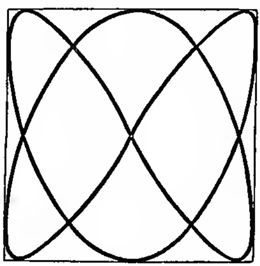

Fig. 5.

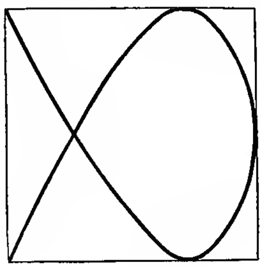

Fig. 6.

the periods are such that two horizontal vibrations occupy the same time as three vertical. Suppose 3τ, 2τ, are the two periods; then the curve in Fig. 5 is defined by the equations

$$x = a \sin \frac{2\pi t}{3\tau}, \qquad y = a \sin \frac{\pi t}{\tau},$$

while in Fig. 6 the first equation is changed into

$$x = a \sin\left(\frac{2\pi t}{3\tau} - \frac{\pi}{6}\right).$$

(The origin is in each case at the centre of the square).

These two curves are particular cases of the appearance presented to the eye by a transparent cylinder having a curve traced upon its surface in the manner above described, that is, in the present case, as follows. Let three complete 'waves,' or periods of the curve $y = a \sin \frac{3x}{2a}$, be drawn upon a rectangular slip of paper, so that the axis of x is parallel to a side of the rectangle; the length of a rectangle which will just contain them will be $4\pi a$, and such a rectangle can therefore be rolled twice round a cylinder of radius a. Suppose this to be done, and the whole to become transparent. Then, if the cylinder be held vertically at a distance from the eye, and turned about its axis, the curve will appear to go through a series of forms of which two are represented in the figures. Fig. 5 is changed into Fig. 6 by a rotation through 30°; a second rotation through 30° brings back Fig. 5, with those parts of the curve in *front* of the cylinder which were at the back, and *vice versâ;* a third similar rotation produces Fig. 6 reversed right and left, and a fourth reproduces the original Fig. 5. Thus a whole cycle of forms is completed in a rotation through 120°, or $\frac{2\pi}{3}$ (see Art. **57**). A greater number of the forms belonging to this and other cases of composition of vibrations may be seen figured in Tyndall's 'Lectures on Sound,' p. 319. They were originally given in the memoir of Lissajous above cited.

61. If the periods of the component vibrations were nearly, but not exactly, in the ratio of 3 to 2, then (as in Art. **58**) the locus of the vibrating point might be represented by the equations

$$x = a \sin\left(\frac{2\pi t}{3\tau} + a'\right), \qquad y = a \sin \frac{\pi t}{\tau},$$

where a' is an angle which varies slowly and uniformly with the time.

In this case the path of the point never actually coincides with any curve corresponding to a constant value of α', just as the path of a disturbed planet is never actually an ellipse; but if the variation of α' be sufficiently slow, the path during one or more repetitions of the period 3τ will be sensibly the same as if α' were constant, and during a longer time will appear to undergo a continuous change through all the forms corresponding to constant values of α', that is, through all the forms presented by the curve on the rotating cylinder.

62. These phænomena may be exhibited by various contrivances, of which the simplest is the kaleidophone. This instrument, invented by Wheatstone and improved by Helmholtz, consists of two thin and narrow rectangular slips of steel, or other elastic material, joined together in such a manner that their longitudinal central axes are in the same straight line, while their planes are at right angles to one another. Thus a compound elastic rod is formed, of which one part can easily vibrate in one direction, and the other part in a direction at right angles to the former. The whole is fixed in an upright position by clamping one of the slips in a stand, and a small bright object (such as a silvered bead) is attached to the top of the other.

If now the rod be disturbed in any manner, so that both its parts are bent, and then left to itself, the motion of the free end will be compounded of vibrations which (when the deflection is not too great) are sensibly rectilinear and harmonic, and at right angles to one another. The period of one component is fixed, depending on the length of the upper part of the rod; but the period of the other component may be altered within certain limits by clamping the lower part at different points.

If the two periods are commensurable, suppose them to be $m\tau$, $n\tau$, where the ratio $m:n$ is in lowest terms. Then $mn\tau$ (the least common multiple of $m\tau$ and $n\tau$) is the period of the resultant vibration, and the path of the free end is a re-entrant curve, described in this period. If then $mn\tau$ be sufficiently small, e.g. not greater than about one-tenth of a second, the impression on the retina made by the bright bead in any position has not time to fade away before the bead comes into the same position again, and the eye sees a continuous bright curve, more or less complicated according as the numbers m and n are greater or smaller.

If the periods of the component vibrations are incommensurable, or if their ratio is expressible only by high numbers, then the real path of the bead is a non-re-entrant curve, or a very complicated re-entrant one. But, in either case, if the ratio is *approximately* expressible by *low numbers*, the actual appearance to the eye will be that of a curve gradually going through all the forms which would be possible if the approximate ratio were exact, in the manner already explained (Arts. **56–59**).

63. Another method, due to M. Lissajous, consists in receiving upon a screen a beam of light which has been successively reflected by two small mirrors fixed to the ends of two tuning-forks vibrating in planes at right angles to one another. It must be observed that the effect in this case depends, not on the motion of translation of the mirrors, but upon their *angular* motion, which, though small, impresses on the reflected beam a sufficient deviation to produce a considerable displacement of the spot of light on a distant screen.

64. The methods described in the last two articles are useful for illustration; but, for the purpose of exact observation, the following, also devised by M. Lissajous, is much better adapted. The object-glass of a microscope is attached to one prong of a tuning-fork, so as to vibrate with it, the other prong being counterpoised: the axis of the object-glass is at right angles to the plane of the vibrations. The eye-piece is fixed. When the fork vibrates, the image of any stationary luminous point, formed by the object-glass, performs also vibrations which are very approximately linear and harmonic; this vibrating image is viewed through the fixed eye-piece, and (the vibrations being sufficiently rapid) appears as a continuous straight line. But, if the luminous point itself, instead of being stationary, performs rectilinear vibrations at right angles to those of the fork, and in the same plane, the image will appear as a curve, either invariable in form, or changing according to the conditions explained in Arts. **55**, &c. Let us suppose, for clearness, that the vibrations of the object-glass are horizontal, and those of the luminous point vertical. Then the horizontal component vibration of the image being harmonic, and of given period, the form of the curve will depend upon the character of the vertical component, and upon the ratio of its period to that of the horizontal component; hence, if the ratio of the periods be known, the character of the vertical vibrations (i. e. the actual vibrations of the luminous point) can be inferred from the observed form of the curve.

In fact, it has been seen that in order that the curve may be distinctly observable, the periods of the two component vibrations must be to one another either exactly, or very nearly, in some simple ratio. The curve may then be supposed to have been formed by rolling on a transparent vertical cylinder a plane curve in which the (horizontal) abscissa is proportional to the time, and the (vertical) ordinate is equal to the actual displacement of the vibrating point, at the given time, from its mean position. We have, therefore, to reproduce this plane curve, which completely defines the actual vibration of the point, by imagining the cylinder to be unrolled. It has been remarked by Helmholtz that it is easier to see what the result of this unrolling would be, when the ratio of the periods is not quite exact, than when it is; because then the cylinder appears to turn about its axis (Art. **57**) so that the observer is enabled to see it, so to speak, on all sides, and to disentangle from one another those parts of the curve which are on the *front* from those which are at the *back*, through the contrary directions of their motion.

65. In reference to this subject, the student will find the following a useful exercise. Draw on a slip of paper two complete waves of a harmonic curve (Art. **66**), choosing the wavelength so that the double wave can just be wound once round a glass cylinder (e.g. a common lamp-chimney). Cut the paper along the curve, so as to obtain a slip of which one edge has the form of the curve; and then, having rolled this on the glass cylinder, mark the glass along the edge of the paper with a glazier's diamond pencil. This is very easily done, and the curve on the glass is distinctly visible both in front and at the back. If now the cylinder be held vertically at a moderate distance from the eye, and turned about its axis, the series of forms will be seen which would be produced if, in an observation of the kind described in the last article, the luminous point performed simple harmonic vibrations with a period nearly equal to half that of the vibrations of the microscope.

Then vary the experiment by substituting for the harmonic curve on the paper either one or more waves of any other kind, and particularly of a zigzag formed by portions of straight lines, thus:

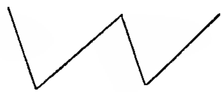

Fig. 7.

The same thing may be done with a wave-length so chosen that the paper will go two, three, or more times round the cylinder. The curve on the glass then becomes more and more complicated, and the marking with the diamond pencil is not so easy, on account of the over-lapping of the paper.

The object of the exercise is to practise the eye in the imaginary unrolling, which is necessary in order to infer the form of the plane curve on the paper from that of the curve as seen on the cylinder. (See Helmholtz, p. 139.)

CHAPTER IV.

THE HARMONIC CURVE.

66. If the motion of a point be compounded of rectilinear harmonic vibrations, and of uniform motion in a straight line at right angles to the direction of those vibrations, the point will describe a plane curve which is called the *harmonic curve.*

Let the straight line be taken for axis of x, and let v be the velocity of the motion along it; then we may suppose the origin to be so taken that the abscissa of the moving point at any time t shall be given by the equation $x = vt$. The ordinate y will be given (Art. 47) by the equation

$$y = a \sin\left(\frac{2\pi t}{\tau} + \alpha\right),$$

where a and τ are the amplitude and period of the harmonic vibrations.

Eliminating t between these two equations, and then putting $v\tau = \lambda$, we obtain

$$y = a \sin\left(\frac{2\pi x}{\lambda} + \alpha\right) \qquad (1)$$

for the equation of the *harmonic curve.* (It was formerly commonly called the 'curve of sines.')

If a wheel were to turn uniformly about a horizontal axis, and at the same time to slide uniformly along it, the projection of any point in the wheel upon a horizontal plane would describe such a curve. Or if a wooden cylinder, terminated at one end by an oblique plane section, were smeared with printer's ink, and then rolled over a sheet of white paper, the line bounding the blackened part of the paper would be a harmonic curve. (The proof of this proposition, which has in fact been already implied (Art. **53**) may be left to the reader.)

If in equation (1) we put $x \pm i\lambda$ instead of x (i being any integer), the value of y is unaltered. The curve, therefore, consists of an infinite series of similar waves, thus,—

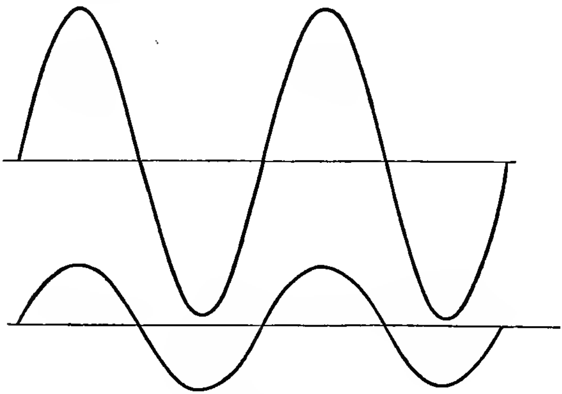

Fig. 1.

which are divided symmetrically into upper and under portions by the axis of x. The distance between corresponding points of two consecutive waves is λ, which is called the *wave-length;* and the constant a, which is the greatest value of the ordinate, is called, as before, the amplitude.

The value of α has no effect on the form of the curve, but determines its *position;* so that a change in α would shift the whole curve along the axis of x.

67. A harmonic curve is most easily drawn in practice by determining a number of points and drawing the curve through them by hand. The points may be found by erecting ordinates at (arbitrary) equal distances, and making them equal to the ordinates of points on a circle of arbitrary radius, which divide the circumference into an arbitrary number of equal parts. The radius of the circle determines the amplitude, and the distance between two consecutive ordinates of the curve, multiplied by the number of parts into which the circle is divided, is the wave-length.

Composition of Harmonic Curves.

68. The formation of a resultant curve, in which the ordinate of any point is the algebraical sum of the corresponding ordinates of the component curves, has been already explained (Art. **10**, &c.). The case in which the component curves are harmonic is specially important.

Two harmonic curves which have equal wave-lengths can always be compounded into another harmonic curve with the same wave-length.

For let the component curves be

$$y = a \sin\left(2\pi\frac{x}{\lambda} + \alpha\right), \qquad y = b \sin\left(2\pi\frac{x}{\lambda} + \beta\right),$$

the value of y in the resultant curve is the sum of the values given by these two equations, and this can be put in the form

$$y = c \sin\left(2\pi\frac{x}{\lambda} + \gamma\right),$$

if c and γ be determined (see Art. **50**) so as to satisfy the equations

$$c\cos\gamma = a\cos\alpha + b\cos\beta,$$
$$c\sin\gamma = a\sin\alpha + b\sin\beta.$$

The value of c, the amplitude of the resultant curve, is

$$\sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)},$$

which may vary from $a + b$ to $a \backsim b$, according to the value $\alpha - \beta$, which determines the relative position of the components.

In the particular case in which the amplitudes of the components are equal, and one of them is half a wave-length before the other, so that $\cos(\alpha - \beta) = -1$, the value of c is 0; or the resultant curve degenerates into a straight line coinciding with the axis of x, the components completely neutralizing one another.

69. It evidently follows that any number of harmonic curves having equal wave-lengths may be compounded into a single harmonic curve with the same wave-length, and of which the greatest possible amplitude is the sum of the amplitudes of the component curves.

70. If the component curves have different wave-lengths, they can no longer be compounded into a single harmonic curve; though, if the wave-lengths be commensurable, the resultant curve is periodic.

Suppose λ to be the least common multiple of the wave-lengths, so that their actual values are $\frac{\lambda}{m}, \frac{\lambda}{n}$, &c., where $m, n, \ldots$ are integers. The equation to the resultant curve is then

$$y = a \sin\left(2\pi \frac{m x}{\lambda} + \alpha\right) + b \sin\left(2\pi \frac{n x}{\lambda} + \beta\right) + \ldots.$$

which does not admit of reduction; but we see that the value of y is unaltered by putting $x + \lambda$ for x, so that the period or wave-length of the resultant curve is λ, that is, the least common multiple of the wave-lengths of the components.

If the component wave-lengths are incommensurable, they have no finite common multiple, so that the period of the resultant curve is infinite: in other words, the resultant is non-periodic.

The *forms* of the component harmonic curves depend only on their amplitudes and wave-lengths; but their *positions* depend upon the constants α, β, &c. A variation in any one of these shifts the corresponding curve along the axis; and any such shifting will evidently alter the form of the resultant curve, and the positions of its points of intersection with the axis, without altering its wave-length.

71. If the wave-length only of the resultant be given, the wave-lengths of the components may be all possible aliquot parts of it, including the *whole* as one case of an aliquot part; and the number of the possible components is therefore unlimited.

Thus every curve which could be constructed in this manner, so as to have a given wave-length λ, would be found amongst those produced by placing along the same axis an unlimited number of harmonic curves as components, with wave-lengths λ, $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, &c.

It is evident that by varying arbitrarily the amplitudes of the components, and shifting them arbitrarily along the axis, an infinite number of resultants could be produced, having all the same wave-length λ. But it could not be assumed without proof that *every possible variety* of periodic curve could be so produced.

This, however (with a limitation to be mentioned below), is true, and constitutes the celebrated theorem of Fourier. Before giving a formal enunciation of it, we will define precisely the meaning of the word *axis* as used above. Corresponding points of a periodic curve lie upon a straight line parallel to a fixed direction. Any straight line parallel to that direction may be called *an axis* of the curve; but it is convenient to call *the axis* that line which cuts off equal areas from the curve on its opposite sides. Thus, the axis of x is *the* axis of a harmonic curve, as defined by the equation $y = a \sin (n x + \alpha)$. This being premised, we proceed to enunciate

Fourier's Theorem.

72. If any arbitrary periodic curve be drawn, having a given wave-length λ, the same curve may always be produced by compounding harmonic curves (in general infinite in number) having the same axis, and having λ, $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, ... for their wave-lengths.

The only limitations to the irregularity of the arbitrary curve are, first, that the ordinate must be always finite; and secondly, that the projection, on the axis, of a point moving so as to describe the curve, must move always in the same direction.

These conditions being satisfied, a wave of the curve may have any form whatever, including any number of straight portions.

Analytically the theorem may be expressed as follows:

It is possible to determine the constants C, C_1, C_2, &c.,

a_1, a_2, &c., so that a wave of the periodic curve defined by the equation

$$y = C + C_1 \sin\left(\frac{2\pi x}{\lambda} + a_1\right) + C_2 \sin\left(2\frac{2\pi x}{\lambda} + a_2\right) + \ldots$$

or
$$y = C + \sum_{i=1}^{i=\infty} C_i \sin\left(\frac{2 i \pi x}{\lambda} + a_i\right),$$

shall have any proposed form, subject to the conditions mentioned above.

By a change of notation, we may write the above equation in the following more convenient form, viz:

$$y = A_0 + 2\sum_{i=1}^{i=\infty} A_i \cos\frac{2 i \pi x}{\lambda} + 2\sum_{i=1}^{i=\infty} B_i \sin\frac{2 i \pi x}{\lambda}. \quad (2)$$

(For a demonstration of the theorem, see the Appendix to this chapter.)

73. The demonstration of the theorem just enunciated includes a determination of the values of the constants. But we may observe here that, assuming the truth of the theorem, we can obtain the expressions for these values at once, by means of the following simple propositions.

If i, j, be integers, all the integrals

$$\int \cos\frac{2 i \pi x}{\lambda} \cos\frac{2 j \pi x}{\lambda} dx, \qquad \int \sin\frac{2 i \pi x}{\lambda} \sin\frac{2 j \pi x}{\lambda} dx,$$
$$\int \sin\frac{2 i \pi x}{\lambda} \cos\frac{2 j \pi x}{\lambda} dx,$$

taken between the limits o and λ, vanish unless $j = i$. If $j = i$ the third integral still vanishes, while the first two have the common value $\frac{\lambda}{2}$, or λ, according as i is different from, or equal to, o.

Hence, if we multiply both sides of the equation (2) by $\cos\frac{2 i \pi x}{\lambda} dx$, or by $\sin\frac{2 i \pi x}{\lambda} dx$, and integrate in each case between the above limits, we obtain, for all values of i, including o,

$$A_i = \frac{1}{\lambda}\int_0^\lambda y \cos\frac{2 i \pi x}{\lambda} dx, \qquad B_i = \frac{1}{\lambda}\int_0^\lambda y \sin\frac{2 i \pi x}{\lambda} dx.$$

74. Thus, whatever the function $f(x)$ may be (the proper limitations being always supposed), the expression

$$\frac{1}{\lambda}\int_0^\lambda f(x)\,dx + \frac{2}{\lambda}\sum_{i=1}^{i=\infty} \cos\frac{2\,i\pi x}{\lambda}\int_0^\lambda f(x)\cos\frac{2\,i\pi.x}{\lambda}\,dx$$

$$+\frac{2}{\lambda}\sum_{i=1}^{i=\infty} \sin\frac{2\,i\pi x}{\lambda}\int_0^\lambda f(x)\sin\frac{2\,i\pi x}{\lambda}\,dx \qquad (3)$$

represents a periodic function of which the value coincides with that of $f(x)$ for all values of x between o and λ, but not for other values, unless the function $f(x)$ be itself periodic, and have λ for its period, so that $f(x+\lambda) = f(x)$; in which last case alone the expression (3) may be taken without limitation as equivalent to $f(x)$.

75. When the actual values of the coefficients A_i, B_i, are required, they have to be found by evaluating the definite integrals (Art. **72**) by which they are expressed.

Suppose, as in Art. **73**, that $y = f(x)$ from $x = 0$ to $x = \lambda$. This equation may subsist in two different senses, which it is important to distinguish.

(1) y may be a given function of x in the ordinary algebraical sense; that is, it may be possible to *assign a rule* by which the value of y can be *calculated for any assumed value of* x. Or (still using the word function in the same sense), y may be a given function from $x = 0$ to $x = a$, another given function from $x = a$ to $x = b$, &c., a, b, &c., being given values between o and λ.

In these cases the ordinary methods of the integral calculus are applicable to the evaluation of the definite integrals. But

(2) y may be a given function of x only in the more general sense (including the former as a particular case) in which the word function should always be understood in mathematical physics; viz. that for every value of x there is a determinate value of y, though it may be impossible to assign any rule for calculating it, in which case it is only to be ascertained by actual observation or measurement.

Thus, if we draw by hand upon paper an arbitrary curve between two points, we can measure as many ordinates as we please, but we can give no rule for calculating the value of y for an assumed value of x.

In such a case the values of the definite integrals can only be found approximately, by measuring a sufficient number of values of y, and applying the method of quadratures. We can thus obtain, by means of equation (2), an *approximate rule* for calculating the value of y for any value of x between given limits. The rule would be *exact* if the coefficients A_i, B_i, were exactly known; but, in order to calculate them *exactly*, we must be in possession of such a rule to begin with.

Since, however, the coefficients actually exist, though we may not be able to ascertain their values rigorously, the abstract truth of the theorem is in no way interfered with; and its great value consists partly in this, that it furnishes an analytical expression (within finite limits) for any function whatever, whether it be a function in the ordinary algebraical sense, or only in the physical sense explained above.

When the coefficients A_i, B_i, are considered as having arbitrary values, the expression (2) (Art. **71**) evidently represents a completely arbitrary periodic function of x.

76. The following is a simple and useful example of the application of the theorem.

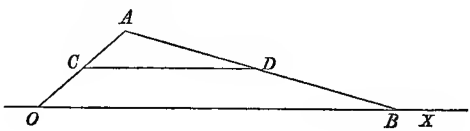

Fig. 1.

Let OA, AB, be two straight lines, cutting the axis of x at the origin O and at a point B, such that $OB = \lambda$.

It is required to determine the coefficients so that the expression (3), Art. **73**, shall give the value of the ordinate at any point of the 'curve' OAB, from $x = 0$ to $x = \lambda$.

Suppose a, b, are the coordinates of A. Then $f(x)$ is $\frac{b}{a}x$ from $x = 0$ to $x = a$, and is $\frac{b}{a-\lambda}(x - \lambda)$ from $x = a$ to

$x = \lambda$. Hence, putting for the present $\frac{2i\pi}{\lambda} = n$, we have (Art. **72**)

$$\lambda A_i = \frac{b}{a}\int_0^a x\cos nx\,dx + \frac{b}{a-\lambda}\int_a^\lambda (x-\lambda)\cos nx\,dx$$

$$\lambda B_i = \frac{b}{a}\int_0^a x\sin nx\,dx + \frac{b}{a-\lambda}\int_a^\lambda (x-\lambda)\sin nx\,dx.$$

Performing the integrations, we find, after easy reductions,

$$A_i = \frac{b}{n^2\lambda a}\cdot\frac{a(1-\cos n\lambda) - \lambda(1-\cos na)}{\lambda - a}$$

$$B_i = \frac{b}{n^2\lambda a}\cdot\frac{\lambda\sin na - a\sin n\lambda}{\lambda - a}.$$

The value of A_0 is most simply obtained directly from the expression $\frac{1}{\lambda}\int_0^\lambda f(x)\,dx$, which gives at once

$$A_0 = \frac{1}{\lambda}\times(\text{area } OAB) = \frac{b}{2}.$$

But it may also be found by evaluating in the usual way the above expression for A_i, when i (and therefore n) $= 0$. Only it is to be observed, that for this purpose we must not assume i to be an integer before putting $i = 0$. The same process gives, of course, $B_0 = 0$. If now we introduce the value of $n\left(=\frac{2i\pi}{\lambda}\right)$ we find, for values of i from 1 to ∞,

$$A_i = \frac{b}{4i^2\pi^2}\cdot\frac{\lambda^2}{a(\lambda-a)}\left(\cos 2i\pi\frac{a}{\lambda} - 1\right)$$

$$B_i = \frac{b}{4i^2\pi^2}\cdot\frac{\lambda^2}{a(\lambda-a)}\sin 2i\pi\frac{a}{\lambda}.$$

These expressions (with $A_0 = \frac{b}{2}$) are to be introduced in the equation (2), Art. **71**, which then becomes the equation to a periodic curve, of which one wave is OAB.

The result is easily reducible to the following form:

$$y = \frac{b}{2} + \frac{b\lambda^2}{\pi^2 a(\lambda-a)}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\sin\frac{i\pi a}{\lambda}\sin\frac{2i\pi\left(x-\frac{a}{2}\right)}{\lambda} \qquad (3)$$

It may be observed that all the periodic terms on the right of (3) vanish for $x = \frac{a}{2}$, and also for $x = \frac{a + \lambda}{2}$; from which it follows that every one of the harmonic curves represented by the several terms passes through the two points C, D, which bisect the lines OA, AB, since the abscissæ of these points are $\frac{a}{2}$, $\frac{a+\lambda}{2}$, and their ordinates are both $= \frac{b}{2}$. It is also evident that the 'axis' (Art. **70**) of the whole locus represented by (3) (consisting of repetitions of OAB) is the line CD indefinitely produced.

77. The following simpler example is also instructive. It is required to find a periodic function, of which the value shall coincide with that of $m\left(x - \frac{\lambda}{2}\right)$ from $x = 0$ to $x = \lambda$. In other words, to find the equation to a periodic curve consisting

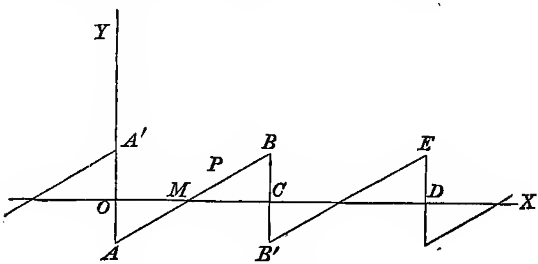

Fig. 2.

of repetitions of the straight line AB (Fig. 2), in which $OC = \lambda$, and $\tan BMC = m$.

In this case we have

$$A_i = \frac{m}{\lambda}\int_0^\lambda \left(x - \frac{\lambda}{2}\right)\cos\frac{2i\pi x}{\lambda}\,dx$$

$$B_i = \frac{m}{\lambda}\int_0^\lambda \left(x - \frac{\lambda}{2}\right)\sin\frac{2i\pi x}{\lambda}\,dx,$$

and it is easily found that $A_i = 0$ for all values of i, while $B_i = \frac{m\lambda}{2i\pi}$. Hence equation (2), Art. **71**, gives,

$$y = -\frac{m\lambda}{\pi}\sum_{i=1}^{i=\infty}\frac{1}{i}\sin\frac{2i\pi x}{\lambda},$$

or

$$y = -\frac{m\lambda}{\pi}\left\{\frac{1}{1}\sin\frac{2\pi x}{\lambda} + \frac{1}{2}\sin 2\frac{2\pi x}{\lambda} + \frac{1}{3}\sin 3\frac{2\pi x}{\lambda} + \ldots\right\} \quad (4)$$

which is the required equation.

Here we may observe that, if the locus be considered as consisting of the detached lines AB, $B'E$, &c., the value of y undergoes a sudden alteration from $m\frac{\lambda}{2}$ to $-m\frac{\lambda}{2}$ when x passes through 0 or any multiple of λ, while for these critical values of x the equation (4) gives $y = 0$, that is, the arithmetic mean of the two values just mentioned (see Appendix to this chapter). On the other hand, since every term in the series (4) is continuous, it is impossible that the sum of the series can undergo an absolutely sudden change of value, without passing through the intermediate values. This subject cannot be fully discussed here; but the true view seems to be that while x varies from a value infinitely near to a critical value, but less, to a value infinitely near, but greater, y passes instantaneously through all values from $m\frac{\lambda}{2}$ to $-m\frac{\lambda}{2}$. Or, geometrically, the locus of the equation (4) ought to be considered as including the portions $A'A$, BB', &c., which are inclined at an infinitely small angle to the true perpendiculars drawn through O, C, &c., and cut them in those points.

Assuming that we may differentiate equation (4) we obtain

$$\frac{dy}{dx} = -2m\left\{\cos\frac{2\pi x}{\lambda} + \cos 2\frac{2\pi x}{\lambda} + \cos 3\frac{2\pi x}{\lambda} + \ldots\right\}$$

Now, considering the series

$$1 + 2(\cos\theta + \cos 2\theta + \cos 3\theta + \ldots)$$

as the limit of

$$1 + 2(c\cos\theta + c^2\cos 2\theta + \ldots),$$

that is, of the fraction $\frac{1 - c^2}{1 - 2c\cos\theta + c^2}$, when c (increasing) becomes $= 1$, we see that it must be taken as representing 0 for all values of θ except the critical values 0, 2π, &c., which give $\cos\theta = 1$, in which case it becomes ∞. And we avoid, as before, the difficulty of attributing a sudden change of value to a series of which all the terms are continuous, by considering that it passes through all values from 0 to ∞, and back again from ∞ to 0, while x varies from being infinitely little less to being infinitely little greater than a critical value. And it follows that the series within brackets in the above expression for $\frac{dy}{dx}$ has in general $-\frac{1}{2}$ for its value, so that $\frac{dy}{dx} = m$; but, while x passes through the critical values 0, λ, &c., the value of the series passes instantaneously through all values from $-\frac{1}{2}$ to ∞ and back again[1].

Hence A', A, B, B', &c. in Fig. 2, are to be considered as points at which the tangent changes its direction, not with absolute suddenness, but by turning round those points. So that if we suppose a moveable point P to be describing the locus ABB', &c., the tangent coincides with AB until P approaches infinitely near to B, and, while P is passing through B, turns through all directions between AB and BB', with which last it coincides as soon as P has passed through B. In other words, the two lines AB, BB', are to be considered not as making an *angle* at B, but as being connected by an infinitely short curved arc, of which the radius of curvature

[1] We purposely avoid here all discussion of the legitimacy of differentiating (4), and of the logical validity of reasoning founded upon the properties of the series (5). What is certain is, that it is impossible to have clear notions of the true nature of Fourier's series, especially in its application to the representation of discontinuous functions, without some such illustrations as those in the text. For a view of the various methods which have been proposed in order to treat the subject with perfect rigour, and of the theoretical questions connected with it, the reader is referred to

Stokes, *On the Critical Values of the Sums of Periodic Series.* (*Camb. Phil. Trans.*, vol. viii.)

De Morgan, *Diff. and Int. Cal.*, p. 605, &c.

Price, *Infinitesimal Calculus*, vol. ii. § 197.

Thomson and Tait, *Nat. Phil.*, vol. i. § 75.

Boole, *On the Analysis of Discontinuous Functions.* (*Trans. of R.I.A.*, vol. xxi. pt. 1.)

But Fourier's original work, *Théorie analytique de la Chaleur*, which unfortunately is now rare, should be consulted by all students who can obtain access to it.

is ∞ at the extremities and infinitely small at the middle. This remark is of course equally applicable to the angular points in the locus, Fig. 1. In general, we may say that Fourier's theorem evades the difficulty of expressing analytically the abrupt changes of value which may, and do, occur in nature, by substituting for them continuous, but infinitely rapid, changes.

78. Any physical condition (such as density, pressure, velocity, &c.) which is *measurable* in magnitude or intensity, and which varies periodically with the time, is expressible as a function of the time by means of Fourier's series. For in the case of actual physical changes, the conditions which make the theorem applicable are necessarily fulfilled. In other words, every actual vibration can be resolved mathematically into harmonic vibrations. If y represent the magnitude in question, and τ the period of its vibration, then y is expressible by an equation of the form

$$y = y_0 + \sum_{i=1}^{i=\infty} C_i \sin\left(\frac{2\,i\,\pi\,t}{\tau} + a_i\right),$$

where y_0 is the *mean*[1] value of y, and each of the variable terms represents by itself a harmonic vibration, of which the period is an aliquot part of the whole period τ.

79. Thus every periodic disturbance of the air, and in particular such vibrations as excite the sensation of sound, can be so resolved. Now we know as a fact that a vibration of the air excites in general the sensation of a musical note, which is not a *simple tone* (Art. **12,** &c.), but a combination of tones corresponding in general to vibrations of which the periods are aliquot parts of the period of the original vibration. (The exceptions to this statement are apparent rather than real. The so-called vibration of a tuning-fork, for example, is not a single 'vibration' in the strict sense of the term, but is compounded of vibrations of which the periods are incommensurable. Hence

[1] The value of y_0, viz. $\frac{1}{\tau}\int_0^\tau y\,dt$, is the average of all the values which y has, at instants separated by infinitely small equidistant intervals, during the period τ.

the whole motion is not really periodic. In this and other similar cases, component tones are heard which do not belong to the harmonic scale of the fundamental tone.)

The ear, therefore, resolves a note into simple tones after the same manner in which Fourier's theorem resolves a vibration into harmonic vibrations; and the question naturally arises whether each simple tone perceived by the ear is really caused exclusively by the corresponding harmonic component of the complex vibration.

We shall soon be able to assign a conclusive reason for believing that this is so; and we shall thus obtain an answer to the question suggested in Art. **14.**

APPENDIX TO CHAPTER IV.

FOURIER'S THEOREM.

THE equation

$$y = \frac{1 - c^2}{1 - 2c\cos(x - a) + c^2}, \tag{1}$$

which may be also written

$$y = \frac{(1 - c^2)\sec^2\dfrac{x-a}{2}}{(1-c)^2 + (1+c)^2\tan^2\dfrac{x-a}{2}},$$

a and c being any constants, represents a periodic curve of which the ordinate is always positive if c be numerically less than 1. In what follows this condition will be supposed, and also that c is *positive*.

Then y will have $\dfrac{1+c}{1-c}$, $\dfrac{1-c}{1+c}$ for maximum and minimum values, corresponding to $x = a \pm 2i\pi$, $x = a \pm (2i + 1)\pi$,

i being any integer; and the distance between the ordinates of corresponding points on successive waves of the curve is 2π.

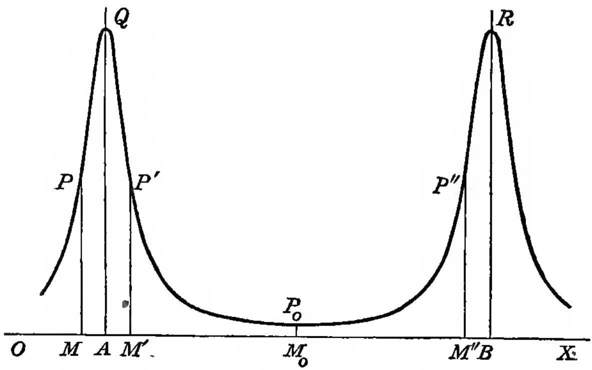

Fig. 3.

Fig. 3 represents a portion of the curve in the case in which $c = \frac{3}{5}$. AQ, BR, are two of the maximum ordinates, and $P_0 M_0$ a minimum ordinate. It is evident that if the value of c be increased, tending towards 1 as a limit, the maximum and minimum ordinates will tend towards ∞ and 0 as limits. At the same time if a fixed point M, or M', be taken, however near to A, the ordinate MP or $M'P'$ will tend to 0 as a limit.

Hence, if the curve be considered as described by a point P, the motion of P tends, as the value of c approaches 1, to become that of a point which moves along OX except when infinitely near to one of the points A, B, &c., but passes those points by going up the left side of the ordinate to an infinite distance, and down again on the right side.

The values $(a \pm 2i\pi)$ of x at A, B, &c., will be called *critical values.*

The area included between the curve, the axis of x, and two consecutive corresponding ordinates (as PM, $P''M''$, or AQ, BR) is 2π. This is easily found by integrating directly the expression $y\,dx$; but it is most simply obtained by first developing y in a series: thus,

$$y = 1 + 2\,(c\cos(x-a) + c^2\cos 2(x-a) + \ldots),$$

from which, observing that

$$\int_x^{x+2\pi} \cos i\,(x-a)\,dx = 0,$$

for all values of i except 0, we find at once

$$\int_x^{x+2\pi} y\,dx = 2\pi.$$

The area between a maximum and next following minimum ordinate, (as $Q\,A\,M_0\,P_0$), is of course $=\pi$. The following are obvious inferences:—

If $M\,P$, $M'\,P'$ be any two fixed ordinates, including one maximum ordinate $A\,Q$ between them, the area $M\,P\,Q\,M'$ tends to 2π as a limit when the value of c approaches to 1. This is true however near either or both ordinates may be to $A\,Q$, so long as neither of them coincides with $A\,Q$ absolutely. Each of the areas $M\,P\,Q\,A$, $A\,Q\,P'\,M'$ tends to become $=\pi$.

And if $M'\,P'$, $M''\,P''$ be any two fixed ordinates, both included between two consecutive maximum ordinates, the area $M'\,P'\,P''\,M''$ tends at the same time to 0. And this is true however near $M'\,P'$, $M''\,P''$ may be to the maximum ordinates $A\,Q$, $B\,R$, so long as there is not absolute coincidence with either.

And, however small the fixed quantities $A\,M$, $A\,M'$, $M''\,B$ may be, it is possible to take c so nearly equal to 1 that the areas $M\,P\,Q\,A$, $A\,Q\,P'\,M'$, $M''\,P''\,R\,B$, shall each differ from π, and the area $M'\,P'\,P''\,M''$ from 0, by quantities less than any assigned quantity.

These conclusions may be expressed analytically as follows. If x_0, x_1, be two values of x, including between them only one critical value (say a), with which neither of them absolutely coincides, and if ϵ, ϵ^1 be any positive constants such that $a+\epsilon$, and $a-\epsilon^1$ are also both included between x_0 and x_1, then the four integrals

$$\int_{x_0}^{a-\epsilon^1} y\,dx, \qquad \int_{a-\epsilon^1}^{a} y\,dx, \qquad \int_a^{a+\epsilon} y\,dx, \qquad \int_{a+\epsilon}^{x_1} y\,dx,$$

have for their limiting values 0, π, π, 0, when c approaches to 1, however small ϵ and ϵ^1 may be. The sum of the four integrals is $\int_{x_0}^{x_1} y\,dx$, and, supposing $x_1 - x_0$ to have the greatest admissible value (2π), this is equal to 2π; and in any case has 2π for its limit when c approaches to 1. The sum of the two middle integrals is $\int_{a-\epsilon^1}^{a+\epsilon} y\,dx$, and this has 2π for its limit.

Let us now suppose that x_0, x_1 have any values such that $x_1 - x_0$ is not $> 2\pi$; and let $f(x)$ be any function which is finite for all values of x from x_0 to x_1 inclusive. Then (y having the value (1) as before) the integral

$$\int_{x_0}^{x_1} f(x) y\, dx, \tag{2}$$

since y is always positive, is equal to the product of the integral $\int_{x_0}^{x_1} y\, dx$, by some quantity intermediate between the (algebraically) least and greatest values which $f(x)$ takes while x varies from x_0 to x_1.

There are three cases to be considered.

(1) There may be no critical value ($a \pm 2i\pi$) from $x = x_0$ to $x = x_1$ inclusively. In this case the limiting value of $\int_{x_0}^{x_1} y\, dx$, and therefore of the integral (2), is 0 when c approaches to 1.

(2) There may be a critical value, say $x = a$, between x_0 and x_1, but not coinciding with either of those limits.

In this case the integral (2) may be considered as the sum of the three integrals

$$\int_{x_0}^{a - \epsilon^1} f(x) y\, dx, \qquad \int_{a - \epsilon^1}^{a + \epsilon} f(x) y\, dx, \qquad \int_{a + \epsilon}^{x_1} f(x) y\, dx,$$

of which the first and third have 0 for their limiting values (by the first case), while the second is equal to $\int_{a - \epsilon^1}^{a + \epsilon} y\, dx$ multiplied by some quantity M intermediate between the least and greatest values which $f(x)$ takes while x varies from $a - \epsilon^1$ to $a + \epsilon$. But ϵ and ϵ^1 may be as small as we please; they may therefore be taken so small that M shall differ infinitely little from $f(a)$. Also, the limiting value of $\int_{a - \epsilon^1}^{a + \epsilon} y\, dx$ is 2π, as was shewn above. Hence we infer that in this case the limit of the integral (2), when c approaches to 1, is

$$2\pi f(a). \tag{3}$$

(3) One or both of the limits, x_0, x_1, may coincide with a critical value. Suppose, for instance, $x_0 = a$, $x_1 = a + 2\pi$. Then by considering the integral (2) as the sum of

$$\int_{x_0}^{x_0 + \epsilon} f(x) y\, dx, \qquad \int_{x_0 + \epsilon}^{x_1 - \epsilon^1} f(x) y\, dx, \qquad \int_{x_1 - \epsilon^1}^{x_1} f(x) y\, dx,$$

it will be seen without difficulty that the limiting value is $\pi f(x_0) + \pi f(x_1)$; that is, $\pi\,(f(a) + f(a + 2\pi))$.

If only one of the limits x_0, x_1, coincided with a critical value a, the result would evidently be $\pi f(a)$.

The above conclusions require modification in the case in which the function $f(x)$ is such as to undergo a sudden finite change of value when x passes through the critical value. If a sudden change take place for any value of x not absolutely coinciding with the critical value, however near to it, the reasoning is not affected, because we may take ϵ and ϵ^1 so small that the value in question shall not lie between $a + \epsilon$ and $a - \epsilon^1$.

But suppose (in case (2)) that $f(x)$ is $= a$ when x is infinitely little less than a, and $= b$ when x is infinitely little greater than a. Then, considering the integral $\int_{a-\epsilon^1}^{a+\epsilon} f(x) y\,dx$, as the sum of $\int_{a-\epsilon^1}^{a} f(x) y\,dx$ and $\int_{a}^{a+\epsilon} f(x) y\,dx$, we see that the first of these will have for its limit $\pi f(a)$, and the second $\pi f(b)$; so that the limiting value of (2) will be

$$\pi\,(f(a) + f(b)),$$

that is, half the sum of the values given by the general rule for the two values of $f(x)$.

We may enunciate these results in the form of the following

Theorem.

If x_0, x_1 be two values of x such that $x_0 < a < x_1$, and $x_1 - x_0$ be not $> 2\pi$, and if $f(x)$ be any function which is finite for all values of x from $x = x_0$ to $x = x_1$ inclusively, then the value of the integral

$$\int_{x_0}^{x_1} f(x) \frac{1 - c^2}{1 - 2c\cos(x - a) + c^2}\,dx, \qquad (2)$$

when c (increasing) approximates indefinitely to 1, has in general for its limit $2\pi f(a)$. But if $f(x)$ undergoes a sudden change of value from a to b when the value of x passes through a, then the limit is $\pi\,(f(a) + f(b))$.

If either $x_0 = a$, or $x_1 = a$, while $x_1 - x_0 < 2\pi$, then the limit is $\pi f(a)$.

But if $x_0 = a$, $x_1 = a + 2\pi$, then it is $\pi\,(f(a) + f(a + 2\pi))$.

On these last suppositions we may provide for the case of sudden changes of value of the function by understanding (if necessary) $f(a)$ to mean $f(a + \epsilon)$, and $f(a + 2\pi)$ to mean $f(a + 2\pi - \epsilon)$, where ϵ is an infinitely small positive quantity.

Since the value of a must not transgress the limits x_0, x_1, and $x_1 - x_0$ must not be $> 2\pi$, we shall give the greatest possible extent to the theorem by supposing $x_1 - x_0 = 2\pi$.

The notation used above was convenient in the course of demonstration; but in actual applications it is better to change it by writing a for x and x for a, so that a becomes the variable in the integration. The principal result may be then stated as follows:

The limit of the integral

$$\int_{a_0}^{a_1} f(a)\, \frac{1 - c^2}{1 - 2c\cos(x - a) + c^2}\, da,$$

where $a_1 - a_0 = 2\pi$, and x has any value between a_0 and a_1, is in general $2\pi f(x)$, when c approximates indefinitely to 1. The special cases are of course to be stated as before, *mutatis mutandis.*

If now in the above integral we substitute for the fraction its development, viz.

$$1 + 2\left(c\cos(x - a) + c^2\cos 2(x - a) + \ldots .\right),$$

we obtain the series

$$\int_{a_0}^{a_1} f(a)\, da + 2\sum_{i=1}^{i=\infty} c^i \int_{a_0}^{a_1} f(a)\cos i(x - a)\, da,$$

and we may therefore affirm, that when c approximates to 1, the limit of this series is in general $2\pi f(x)$ for any value of x between a_0 and a_1, but is $\pi(a + b)$ for a value of x corresponding to a sudden change from a to b in the value of $f(x)$, and $\pi\left(f(a_0) + f(a_1)\right)$ for $x = a_0$ or $x = a_1$.

Assuming, then, that when $c = 1$ the value of the series becomes equal to its limit[1], we may write the result as follows (excluding of course the exceptional cases):

$$2\pi f(x) = \sum_{i=-\infty}^{i=\infty} \int_{a_0}^{a_1} f(a)\cos i(x - a)\, da,$$

for values of x between a_0 and a_1.

If now we write $\dfrac{2\pi x'}{\lambda}$ for x, and $\dfrac{2\pi a'}{\lambda}$ for a, assuming also $a_0 = 0$, $a_1 = 2\pi$, the above equation becomes

$$f\left(\frac{2\pi x'}{\lambda}\right) = \frac{1}{\lambda}\sum_{i=-\infty}^{i=\infty} \int_0^{\lambda} f\left(\frac{2\pi a'}{\lambda}\right)\cos\frac{2i\pi(x' - a')}{\lambda}\, da';$$

[1] This assumption appears to be the only point in the demonstration which is open to objection. But we cannot here discuss the proposition, 'what is true up to the limit is true in the limit.' On the convergence of the series (3), see the demonstrations of Tait and Thomson and of Stokes, referred to above (Art. 77).

or, omitting accents, and writing $f(x)$ instead of $f\left(\frac{2\pi x}{\lambda}\right)$,

$$f(x) = \frac{1}{\lambda}\sum_{i=-\infty}^{i=\infty}\int_0^\lambda f(a)\cos\frac{2i\pi(x-a)}{\lambda}da,$$

which is now true for values of x between 0 and λ.

This equation may, by an obvious transformation, be written thus:

$$f(x) = \frac{1}{\lambda}\int_0^\lambda f(a)\,da + \frac{2}{\lambda}\sum_{i=1}^{i=\infty}\cos\frac{2i\pi x}{\lambda}\int_0^\lambda f(a)\cos\frac{2i\pi a}{\lambda}da$$
$$+\frac{2}{\lambda}\sum_{i=1}^{i=\infty}\sin\frac{2i\pi x}{\lambda}\int_0^\lambda f(a)\sin\frac{2i\pi a}{\lambda}da, \quad (3)$$

in which form it is usually most convenient.

A further transformation, which gives an expression for $f(x)$ by means of a double definite integration, and which is also often referred to as 'Fourier's Theorem,' is not required for the purposes of this treatise.

If we had taken above $a_0 = -\pi$, $a_1 = +\pi$, we should have found in the same way a series only differing from (3) in having $-\frac{\lambda}{2}$, $+\frac{\lambda}{2}$ for limits in the integrations, instead of 0, λ. Suppose this alteration made, and then suppose further that $f(x)$ is an *odd* function, that is, that $f(-x) = -f(x)$. In this case it is easily seen that the integral $\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(a)\cos\frac{2i\pi a}{\lambda}da$ vanishes for all values of i, including $i = 0$, since it may be divided into pairs of equal elements, but with opposite signs. But

$$\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} f(a)\sin\frac{2i\pi a}{\lambda}da = 2\int_0^{\frac{\lambda}{2}} f(a)\sin\frac{2i\pi a}{\lambda}da$$

since the integral on the left may be divided into pairs of equal elements with the same sign.

Hence, if we put $\frac{\lambda}{2} = l$, we obtain instead of (3)

$$f(x) = \frac{2}{l}\sum_{i=1}^{i=\infty}\sin\frac{i\pi x}{l}\int_0^l f(a)\sin\frac{i\pi a}{l}da, \quad (4)$$

which is true from $x = -l$ to $x = +l$ for any function $f(x)$ which is odd between those limits. And it will evidently be true from $x = 0$ to $x = l$ for *any* function.

If $f(x)$ besides being odd, is periodic, with period (or wave-length) $2l$, then (4) will be true without limitation.

The case in which $f(x)$ is an *even* function ($f(x) = f(-x)$) may be left to the reader.

CHAPTER V.

VIBRATIONS OF AN ELASTIC STRING.

80. Of the various modes in which musical sounds may be produced, one of the most usual in practice, and simple in theory, consists in the vibration of elastic strings.

The term 'elastic string' is to be understood as implying ideal qualities, which do not belong absolutely to any actual string; namely, non-resistance to flexure, and extensibility according to the law that extension is proportional to tension.

According to this law, if l be the natural length of the string (or its length when not subjected to any tension), and l' the length when it is stretched by a force T, then

$$l' - l = \frac{T}{E} l,$$

where E is a constant, of which the value depends on the nature of the string.

Since the homogeneity of the above equation requires that $\frac{T}{E}$ should be an abstract number, E is a quantity of the same kind as T, that is, a *force*. And since the supposition $l' = 2\,l$ gives $E = T$, we see that E may be defined as the force which would be required to stretch the string to twice its natural length.

No actual string is perfectly flexible, nor indefinitely extensible, according to the above law. But when the changes of length are small, it is proved by experiment that they are sensibly proportional to the changes of tension. And when the tension is considerable, and the thickness of the string very small in comparison with its length, the resistance to flexure is very small in comparison with the resistance to extension.

Under these conditions the phænomena with which we are concerned are so nearly the same as they would be in the case of the ideal string, that for most purposes they may be considered identical. The case in which imperfect flexibility must be taken into account will be considered apart.

81. Supposing the two ends of an elastic string to be fixed at points of which the distance from one another is greater than the natural length of the string, its form in the condition of equilibrium will be a straight line, and its tension will be the same at all points[1]; namely, that with which the string would have to be pulled at any point in order to maintain the equilibrium, if it were cut there.

Suppose now the string to be slightly disturbed by any forces whatever, which, at a certain instant, cease to act. The subsequent motion of the string will depend upon the positions and velocities of its particles at that instant; and the mechanical theory shews that it will *vibrate*, that is, will go through a series of periodic changes, which, theoretically, would never cease. The vibration actually does cease because the string gradually gives up its motion to the surrounding air, to the bodies which sustain the tension of its ends, and, in a different form, to its own molecules.

The condition of the string at the instant when the disturbing forces cease to act is called its *initial condition.*

When the initial displacements of all the particles of the string are *lateral*, that is, at right angles to its original direction, and when the initial velocities are lateral also, the vibrations are sensibly lateral.

When the initial displacements and velocities are longitudinal, the vibrations are necessarily longitudinal, and the form of the string remains a straight line.

Vibrations of both kinds can coexist, but it is best to consider them separately. Longitudinal vibrations of a string are

[1] The effect of gravity is neglected, being insensible in all cases of the kind here considered.

of the same kind as those of a straight rod, which will be treated of in another chapter. In the case of a string they are practically unimportant. At present, therefore, we shall only examine the lateral vibrations.

82. Lateral initial displacements and velocities, if they are not in one plane, may be resolved into components in two arbitrary planes at right angles to one another; and each set of components gives rise to vibrations in its own plane, which coexist independently, and of which the actual vibration is the resultant.

It is sufficient, therefore, to suppose that the initial displacements and velocities, and therefore the subsequent vibrations, are in one plane.

83. The true nature of the vibrations of a finite string may be best understood by considering them under that aspect which is suggested by the dynamical theory (see Chap. VII); in which the string is regarded as infinitely long, and out of the various possible forms of motion of an infinitely long string, those are selected which are characterised by the existence of *nodes*, that is, of motionless points. Any two such motionless points of the string might evidently become *fixed* without disturbing the motion; and then all the string not included between them might be removed, so as to leave a finite string, with fixed ends, which would continue to have the same motion as it would have had under the original conditions. We shall therefore begin with the case of an infinite string.

84. It must be always understood that though the following propositions, in so far as they are merely geometrical, are true without limitation, it is only when the displacements are infinitely small that they are rigorously true mechanically; but they are sensibly true within limits wide enough for the most important practical applications.

85. The simplest form of motion of an infinitely long elastic string consists in the *transmission of a single wave.*

Let AB represent part of the string in its undisturbed condition, and let CD be a line parallel to AB, of which any arbitrary portion QR is bent into an arbitrary curve. Imagine CD

to be moving with a constant velocity, in the direction of its length, either towards the right or towards the left, and that part of the string AB which at any time is opposite to QR, to

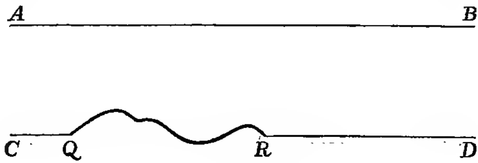

Fig. 1.

be always bent into the same form by a lateral displacement of its particles, the rest being always straight. Then the string AB will be *transmitting a wave.* When a single wave is thus transmitted, any particular particle of the string is disturbed during the passage of the wave, and is at rest at all times before and afterwards.

86. It is necessary, however, to explain how such a wave could originate.

Suppose a portion of the string, which we may call PQ, to be bent into any arbitrary form, and a determinate lateral velocity to be then communicated to each particle of PQ, according to any arbitrary law consistent with the continuity of the string. The subsequent motion will depend on the form of PQ, and on these initial velocities of its particles.

Now the form of PQ being given arbitrarily, it is possible to assign the initial velocities in one way so that a single wave of the form PQ shall be transmitted to the right, and in another way so that a similar wave shall be transmitted to the left.

But if the initial velocities were assigned arbitrarily, the wave PQ would in general be resolved into two components, of which one would travel to the right, and the other to the left.

At present, however, we assume that the initial velocities have been such as to give rise to a single wave.

Any number of similar or dissimilar waves may be transmitted at the same time, either in the same or in contrary directions. It is convenient to distinguish the direction of transmission of waves, by calling them positive or negative according as they are transmitted from left to right, or from right to left.

87. The velocity of transmission depends only on the nature and tension of the string, and not on the form or length of the wave.

88. Let us now suppose two waves of equal length, but otherwise of arbitrary forms, to be transmitted in contrary directions so as to meet. After passing one another they will proceed in their original forms. But during the passage a part of the string will be disturbed by both waves at once, and the displacements of its particles at any instant will be the sums of the displacements due to the separate waves. This part of the string will evidently have for its length the length of a wave, and for its middle point the point at which the ends of the waves first meet.

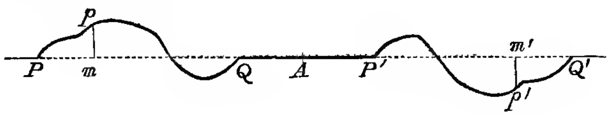

Fig. 2.

Let PQ, $P'Q'$ (Fig. 2) represent the positive and negative waves before meeting, and A the point at which they will first meet. And let us now further suppose that each wave is (as in the Figure) a *reversed copy of the other*, so that the figure would not be altered by turning the paper upside down. (We may express the same condition by saying that every straight line through A which cuts one wave, cuts the other also at a point equidistant from A.) Two such waves may be called *contrary waves.*

Now let pm, $p'm'$ be any two ordinates equidistant from A. The corresponding points p, p' of the two waves will arrive opposite to A at the same instant, and the displacement of A at that instant will be the algebraical sum of pm and $p'm'$, that is zero, since pm, $p'm'$ are equal in length, but opposite in direction. Hence, since the same thing is true for every pair of corresponding points, the point A will not be displaced at all; in other words, the string will have a *node* at A.

Fig. 3.

Thus, when an infinite string transmits two contrary waves in opposite directions, there is a node at the point at which they first meet.

89. Let us now inquire what conditions are necessary in order that there may be *two* nodes. Let A be the place of the first node and B of the second; and suppose that a positive wave PA, and a contrary negative wave AQ, are just beginning to meet at A. The wave PA, passing on to the right, will after a certain time arrive at the position $P'B$, and will then begin to disturb B unless it meets at that instant a contrary wave BQ' moving towards the left. But this new negative wave BQ', when it arrives at the position AQ, will begin to disturb A unless it meets a new positive wave. And, continuing the same reasoning, we see that there must be an infinite series of positive waves meeting an infinite series of negative contrary waves.

There will then be nodes at A and at B; but it is evident that there will be also an infinite number of other nodes at equal intervals. Hence an infinite string cannot have more than one node without having an infinite number.

The distance between corresponding points of consecutive positive (or negative) waves is evidently twice AB. For when the positive wave which at any instant is at PA shall have arrived at $P'B$, the next following one will not be at PA, but so far behind that position as to meet the negative wave BQ' when the latter arrives at AQ.

90. Let us, however, fix our attention upon the portion of string included between two consecutive nodes A, B, supposing the rest of it to be hidden from view, and consider its condition at the instant when the positive and negative waves have just passed one another at A. The form of the visible string will then be this,—

Fig. 4.

and the positive wave, of which the left-hand end is now at A, will be transmitted unaltered until its right-hand end reaches B. It then begins to meet the negative wave coming from the invisible part of the string, and to be compounded with it; by this composition the disturbed part of the visible string undergoes a change of form, until the positive wave has passed completely out of sight, and is replaced by the negative wave, thus,—

Fig. 5.

which is transmitted to A, where it meets a new positive wave, and a converse series of changes takes place; and so on indefinitely. Thus, a wave will appear to pass backwards and forwards between A and B, and to be *reflected* at those points. Each reflection consists in a shortening of the wave to half its length, followed by an equal lengthening; and during this shortening and lengthening its form is changed into that of the contrary wave.

91. During motion of this kind, both the whole string and the portion AB are in a state of *vibration*, that is, undergoing a periodic change of condition; and the *period* of a vibration, or the interval of time between the recurrence of similar conditions, is evidently the time required for the transmission of a wave through twice the distance (AB) between consecutive nodes.

92. Hitherto we have considered a wave as consisting only of the bent or curved portion of the string. But we will now call the whole portion included between two consecutive similar points a *wave*, as we have always done before in treating of periodic curves, so that a wave-length is now twice the distance between consecutive nodes. In the preceding illustrations we have supposed only a small portion of this wave to be curved, in order to make the process of reflection clearly intelligible; but since the length attributed to the curved portion was arbitrary, we may suppose it to occupy, as it generally does in fact, the whole wave-length.

Let AC be a wave-length, the figure representing part of a series of positive waves, and also of the contrary negative waves, in the position which they have at a given instant. The

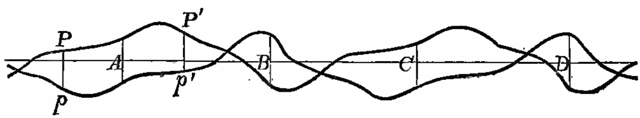

Fig. 6.

actual form of the string at that instant (which is not drawn) would be found by compounding the two curves. Thus the middle point of any ordinate Pp would be a point on the string.

The *nodes* are the points A, B, C, D, &c. on the axis, which at this instant, as at all others, bisect the ordinates drawn through them. Now suppose Pp, $P'p'$ are any two ordinates equidistant from any node A; it is evident that the middle points of these ordinates lie upon a straight line which is bisected at A; in other words, *every straight line drawn through a node at any instant and terminated both ways by the string, is bisected by the node.*

93. Hence, considering a whole wave-length of the resultant curve, occupying two nodal intervals, we see that the half wave on one side of the middle node is always *contrary* in form (Art. **88**) to the other half.

Thus the whole infinite string will always have the form of a series of similar waves divided into contrary halves by alternate nodes; and the two halves will have exchanged their forms after every half period.

The string will therefore appear to *oscillate*, but there will be in general no visible appearance of *transmission*[1] of waves in either direction, the positive and negative series completely disguising each other except in the case in which a considerable portion of each wave is *straight:* in this case only, the curved portions will appear to be transmitted in contrary directions; or, if only a portion of string between two consecutive nodes be looked at, to be reflected backwards and forwards from node to node in the manner explained in Art. **90**.

94. It has been already explained that, in order to pass from an infinite to a finite string, we must suppose two nodes to become fixed. But these may, of course, include any number of nodes between them. Hence the most general form of the vibration of a finite string, fixed at both ends, consists in an oscillatory motion, with nodes dividing the string into equal aliquot parts. The form of the part included between any two consecutive nodes will always be *contrary* to that of the adjacent intervals; and any two adjacent intervals exchange forms after half a period. The wave-length is twice the distance between consecutive nodes; and the period of a vibration is the time occupied by the transmission of a wave over this double distance. Hence the period is known if we know the length of the string, the number of nodes, and the velocity of transmission.

When there are no nodes between the extreme points, the wave-length is twice the length of the string, and the period is the time of transmission over this double length. This is called simply 'the time of vibration' of the string.

[1] Hence such waves are often called *stationary*.

If l be the length of the (stretched) string, W its weight, T the tension by which it is stretched (expressed in terms of the same unit as W), and g the so-called accelerating force of gravity (that is, the velocity acquired by a falling body at the end of a unit of time), then the velocity of transmission of a lateral wave is $\sqrt{\frac{l g T}{W}}$, and the time of a vibration is therefore $2\sqrt{\frac{l W}{g T}}$ (see below, Art. **123**).

95. It may be here said, once for all, that the term 'vibration' is always to be understood as implying a *complete cycle of changes*.

In many of the most usual cases (as in that of a string) the vibration may be divided into two parts, equal in duration and converse in character; and it has been the practice of some (especially of French) writers to call each half a *single vibration*, and the two together a *double vibration;* or, what is much worse, to use the term vibration without distinctly explaining in which sense it is to be understood.

In the case of a common pendulum the habit of giving the name *vibration* to what is only half a cycle, namely, a swing in one direction, has become inveterate.

In this work, however, the term will always mean what Prof. De Morgan has proposed to call a *swing-swang*.

Thus, the time of vibration of a so-called seconds' pendulum is *two seconds*.

It must be remembered also that a vibration in general is not necessarily divisible into a 'swing' and a 'swang' of equal duration and opposite character.

96. We will now proceed to the analytical expression of the results obtained in the preceding articles.

If $f(x)$ be any function which has one real and finite value for every value of x, the equation $y = f(x - vt)$ will represent the form of a string which is always bent so as to follow that of a curve $y = f(x)$, supposing the latter to be moving in the positive direction of the x-axis with a constant velocity v. (For if in the former equation we remove the origin along the axis to a distance vt, the equation becomes $y = f(x)$.)

Similarly $y = F(x + vt)$ represents the transmission of the form $y = F(x)$, in the negative direction, with an equal velocity. Hence the equation

$$y = f(x - vt) + F(x + vt)$$

represents the form of a string in which the displacement of any particle at the time t is the sum of the displacements due to both these causes. And if $f(x)$, $F(x)$ are both periodic functions, the last equation will represent the transmission of two sets of waves in opposite directions.

Supposing, then, these periodic functions to be expressed by means of Fourier's series (Art. 71), and to have the same wave-length $2l$, the equation will take the form

$$y = \sum_{i=1}^{i=\infty} C_i \sin\left(\frac{i\pi(x - vt)}{l} + a_i\right) + \sum_{i=1}^{i=\infty} C'_i \sin\left(\frac{i\pi(x + vt)}{l} + a'_i\right).$$

(The constant term is omitted, because it could be got rid of, if necessary, by removing the origin along the axis of y.)

97. We have not yet introduced the condition of the existence of *nodes*. Let us now suppose that there is a node at the origin; that is, that when $x = 0$, $y = 0$ for all values of t. When $x = 0$, the terms of the order i in the above series may be written

$$\sin\frac{i\pi vt}{l}(-C_i \cos a_i + C'_i \cos a'_i)$$
$$+ \cos\frac{i\pi vt}{l}(C_i \sin a_i + C'_i \sin a'_i),$$

and it is evident that the coefficient of each must vanish separately; that is,

$$C_i \cos a_i - C'_i \cos a'_i = 0,$$
$$C_i \sin a_i + C'_i \sin a'_i = 0.$$

These equations are satisfied by assuming $C'_i = C_i$, $a'_i = -a_i$; and if we introduce these conditions in the value of y at the end of Art. 96, and put $2C_i \cos a_i = A_i$, $2C_i \sin a_i = B_i$, $2l = v\tau$, we obtain

$$y = \sum_{i=1}^{i=\infty} \sin\frac{i\pi x}{l}\left(A_i \cos\frac{2i\pi t}{\tau} + B_i \sin\frac{2i\pi t}{\tau}\right), \qquad (1)$$

in which τ is the period of vibration. Since y vanishes not only when $x = 0$, but also when x is any multiple of l, for all values of t, we see that the condition of the existence of a node is satisfied not only at the origin, but also at an infinite number of points separated from one another by an interval l (or half a wave-length).

Hence this equation represents in the most general manner the oscillatory motion of an infinite string described in Art. **93**. And if we confine our attention to values of x between 0 and l, it represents the vibration of a finite string of length l, fixed at its two ends.

98. If the initial form of the finite string, and the initial velocities of its particles, are given, the values of the constants A_i, B_i are determined. Suppose, for instance, that when $t = 0$, y is to be a given function $f(x)$ from $x = 0$ to $x = l$, and $\frac{dy}{dt}$ another given function $\phi(x)$ within the same limits. Putting, then, $t = 0$ in the equation (1), and in its differential coefficient with respect to t, we must have

$$\sum_{i=a}^{i=\infty} A_i \sin\frac{i\pi x}{l} = f(x),$$

$$\sum_{i=1}^{i=\infty} i B_i \sin\frac{i\pi x}{l} = \frac{\tau}{2\pi}\phi(x),$$

from $x = 0$ to $x = l$. Hence, multiplying each side of these equations by $\sin\frac{i\pi x}{l}dx$, and integrating from $x = 0$ to $x = l$, we obtain

$$A_i = \frac{2}{l}\int_0^l f(x)\sin\frac{i\pi x}{l}dx, \qquad B_i = \frac{\tau}{i\pi l}\int_0^l \phi(x)\sin\frac{i\pi x}{l}dx.$$

(It should be observed that $f(x)$ is necessarily expressible by means of a series of *sines* only (see Appendix to Chap. IV, *ad fin.*); for the form of the string is at any time half a wave of a series in which the alternate values are *contrary* (Art. **93**), so that the function $f(x)$, besides satisfying the conditions $f(ml) = 0$ for all integer values of m including 0, must also satisfy the conditions $f(x + 2ml) = f(x)$ and $f(2ml - x) = -f(x)$; and these are satisfied by every such term as $\sin\frac{i\pi x}{l}$, but not by $\cos \frac{i\pi x}{l}$. The same remark applies to $\phi(x)$, which, as is easily seen *à priori*, ought to satisfy the same conditions.)

A case frequently useful is that in which the string is initially at rest, and its initial form is given by an equation

$$y = \sum_{i=1}^{i=\infty} C_i \sin \frac{i\pi x}{l}.$$

In this case $\phi(x) = 0$, and $f(x)$ is the series just given for y; and therefore $B_i = 0$, and $A_i = \frac{2}{l}\int_0^l y \sin \frac{i\pi x}{l} dx = C_i$; hence the subsequent vibration is given by the equation

$$y = \sum_{i=1}^{=\infty} C_i \sin \frac{i\pi x}{l} \cos \frac{2 i\pi t}{\tau},$$

where $\tau = \frac{2l}{v}$ as before.

99. Another useful case is that in which the initial form of the string is a bent line AQB, where $AB = l$, and a, b are the coordinates of Q reckoned from A.

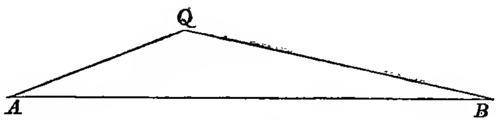

Fig. 7.

Suppose further, that the initial velocities of all points are zero. This is the case of a string made to vibrate by being pulled aside at one point and then left to itself.

Here $\phi(x)$ is 0, and $f(x)$ is $\frac{b}{a}x$ from $x = 0$ to $x = a$, and $\frac{b}{a-l}(x-l)$ from $x = a$ to $x = l$; and therefore $B_i = 0$, and

$$A_i = \frac{2b}{l}\left\{\int_0^a \frac{x}{a} \sin \frac{i\pi x}{l} dx + \frac{1}{a-l}\int_a^l (x-l) \sin \frac{i\pi x}{l} dx\right\}$$

$$= \frac{2bl^2}{\pi^2 a(l-a)} \cdot \frac{\sin \frac{i\pi a}{l}}{i^2}.$$

(The reader should observe that the problem here solved differs from that in Art. **76** in this respect, that the bent line is here half a wave, whereas there it was a whole wave.)

Introducing this value of A_i in equation (1), we obtain

$$y = \frac{2\,b\,l^2}{\pi^2\,a\,(l-a)} \sum_{i=1}^{i=\infty} \frac{\sin \dfrac{i\,\pi\,a}{l}}{i^2} \sin \frac{i\,\pi\,x}{l} \cos \frac{2\,i\,\pi\,t}{\tau}, \qquad (2)$$

which gives the form of the vibrating string at the time t. (On a particular case of this formula see below, Art. **118**.)

100. It is easy to trace geometrically the variations of form during a vibration. The equation

$$y = f(x - vt) + F(x + vt),$$

and its differential coefficient with respect to t,

$$\frac{dy}{dt} = -\,v f'(x - vt) + v\,F'(x + vt)$$

give, when $t = 0$,

$$y = f(x) + F(x)$$

$$\frac{dy}{dt} = -\,v\,(f'(x) - F'(x)),$$

and these expressions must coincide, from $x = 0$ to $x = l$, with the given functions which define the initial positions and velocities. Hence, supposing the initial value of y to be $\phi(x)$, and of $\frac{dy}{dt}$ to be $v\,\psi'(x)$, we have

$$f(x) + F(x) = \phi(x),$$

$$f'(x) - F'(x) = -\,\psi'(x),$$

whence $$f(x) - F(x) = C - \psi(x).$$

These equations give (the arbitrary C being taken $= 0$)

$$f(x) = \tfrac{1}{2}(\phi(x) - \psi(x)), \qquad F(x) = \tfrac{1}{2}(\phi(x) + \psi(x)),$$

so that $f(x)$ and $F(x)$ are known from $x = 0$ to $x = l$. That is, one half of the positive and one half of the negative wave are given in the position in which they produce by composition the initial form of the string. And since, in order to maintain the nodes at its extremities, the half of the negative wave to the right of the string must be *contrary* in form (Art. **88**) to the given half of the positive wave, while the half of the positive wave to the left of the string must be contrary to the given half of the negative wave, a whole wave of each is determined. And we have only to shift the positive wave to the right and the negative to the left, through equal distances, and compound those parts which fall within the limits of the string, in order to find the form at the time corresponding to that amount of shifting.

101. The case treated in Art. **99** affords an easy and interesting example. The initial velocities being zero, we have $\psi'(x) = 0$, and we may therefore take $\psi(x) = 0$; hence

$$f(x) = F(x) = \tfrac{1}{2}\phi(x);$$

that is, the given halves of the positive and negative wave coincide, and the ordinate at any point of each is half the ordinate of the corresponding point of the string in its initial form. Hence, completing the waves as in the last article, we have the following forms and initial positions:

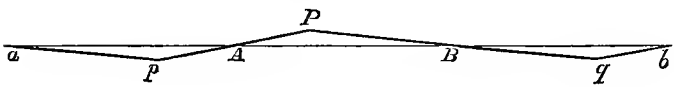

Fig. 8.

APB represents the halves of the positive and negative waves, coinciding when $t = 0$. apA is the other half of the positive, and Bqb of the negative wave. A, B are, as before, the ends of the string, and the initial form is that given by compounding the two 'curves' which coincide in APB; that is, by doubling the ordinate at every point.

After the lapse of a quarter of a period, the waves will have been shifted (one to the right and the other to the left) through a distance equal to half the length (AB) of the string; and that part of the figure which is within the limits AB will have been changed into the following:

Fig. 9.

And from this we obtain by composition the following as the form of the string at that instant:

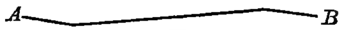

Fig. 10.

At the end of half a period the form will be evidently contrary to the initial form, the two half waves again coinciding; and at the end of three-quarters of a period, the form will be that of the last figure reversed right and left (as it would be seen through the paper from the back).

If the original point of flexure be at the middle of the string, the form at the end of the first, third, &c. quarter of a period will be a straight line. But in other cases the string will always be divided into either two or three straight portions, viz. three in general, but only two at the instants when the positive and negative waves coincide, and also at the instants when a summit of each is opposite the same end of the string,—in all, four times in each vibration.

The value of $\frac{dy}{dx}$ at either end of the string varies discontinuously in a remarkable way. This may be traced geometrically without difficulty, but the following mode of proof affords a good example of the use of Fourier's series.

Putting $x = 0$ in the value of $\frac{dy}{dx}$ derived from (2), we have, at that end of the string,

$$\frac{dy}{dx} = \frac{2\,b\,l}{\pi\,a\,(l-a)} \sum_{i=1}^{i=\infty} \frac{1}{i} \sin\frac{i\,\pi\,a}{l} \cos\frac{2\,i\,\pi\,t}{\tau}. \qquad (3)$$

Now the series $\sum_{i=1}^{i=\infty} C_i \cos\frac{2\,i\,\pi\,t}{\tau}$ evidently represents a periodic function of which the period is τ, and which satisfies the condition $f(t) = f(\tau - t)$; and any such function can, with the addition if necessary of a constant term, be represented by such a series. We may include the constant term by extending the summation to $i = 0$.

Consider, then, a function $f(t)$ of which the value is

$$\begin{aligned} &c \text{ from } t = 0 \text{ to } t = a; \\ &k \text{ from } t = a \text{ to } t = \tau - a; \\ &c \text{ from } t = \tau - a \text{ to } t = \tau. \end{aligned}$$

Assuming $f(t) = \sum_{i=0}^{i=\infty} C_i \cos \frac{2 i \pi t}{\tau}$, we shall have

$$C_0 = \frac{1}{\tau}\int_0^\tau f(t)\, dt,$$

and

$$C_i = \frac{2}{\tau}\int_0^\tau f(t) \cos \frac{2 i \pi t}{\tau}\, dt,$$

for all values of i except 0. Performing the integrations in separate parts (as in finding A_i in (2)), we find

$$C_0 = k + \frac{2a}{\tau}(c - k),$$

$$C_i = \frac{2(c-k)}{i\pi} \sin \frac{2 i \pi a}{\tau} \text{ when } i \text{ is not } 0.$$

Hence the discontinuous function $f(t)$ is represented by the series

$$k + \frac{2a}{\tau}(c-k) + \frac{2(c-k)}{\pi} \sum_1^\infty \frac{1}{i} \sin \frac{2 i \pi a}{\tau} \cos \frac{2 i \pi t}{\tau}.$$

which will be identical with (3) if we assume a, k, and c so that

$$k + \frac{2a}{\tau}(c-k) = 0, \qquad \frac{2a}{\tau} = \frac{a}{l}, \qquad c - k = \frac{b l}{a(l-a)},$$

from which we find

$$c = \frac{b}{a}, \qquad k = -\frac{b}{l-a}, \qquad a = \frac{a\tau}{2l};$$

and it follows that the string, at the point A (Fig. 7, Art. **98**), maintains its initial direction AQ from $t = 0$ to $t = \frac{a\tau}{2l}$ (that is, during the time of wave-transmission along a distance a), it then suddenly becomes parallel to QB and maintains that direction until $t = \left(1 - \frac{a}{2l}\right)\tau$, when it resumes the first direction until the end of the period τ, and so on successively. The representation of these changes by a diagram in which the abscissa and ordinate are proportional to t and to the value of $\frac{dy}{dx}$ at A (or $f(t)$) is obvious, and may be left to the reader.

(See Helmholtz, p. 93, where a greater number of intermediate forms of the string is given, and a diagram representing the values of $\frac{dy}{dx}$ at one end, which is taken as the curve of pressure (or rather of variation of pressure) on a bridge supposed to support the string at that end. The variations of pressure are sensibly proportional to those of $\frac{dy}{dx}$, as will be seen in Chap. VII.)

CHAPTER VI.

VIBRATIONS OF A STRING (*continued*).

102. The most general form of the infinitesimal vibrations (in one plane) of a string is given by the equation (1) of Art. **97**, which, by a transformation inverse to that employed in Art. **72**, may be written thus:

$$y = \sum_{i=1}^{i=\infty} C_i \sin \frac{i\pi x}{l} \cos\left(\frac{2i\pi t}{\tau} + a_i\right), \qquad (13)$$

where x is the distance of any point in the string from one end, and y is the lateral displacement of that point at the time t. This displacement is therefore the sum of the displacements, in general infinite in number, represented by the several terms of the series; and the vibration of the whole string may accordingly be said to be compounded of the vibrations represented by those terms.

Let us then consider separately the vibration represented by the term of the order i. Supposing this term to exist alone, we have, instead of (13),

$$y = C_i \sin \frac{i\pi x}{l} \cos\left(\frac{2i\pi t}{\tau} + a_i\right),$$

and the form of the string at any time is therefore a harmonic curve cutting the axis in fixed points or nodes, which divide the whole length into i equal parts, while the amplitudes of the waves of the curve vary periodically with the time, and every individual point (except the nodes) performs harmonic vibrations with the same period $\frac{\tau}{i}$. Thus, the motion of the whole string is an oscillation with nodes, of the kind described in

Art. **93**, but with this distinction, that the waves are of the harmonic form.

Now if, returning to the general equation (13), we suppose all the coefficients preceding C_i to vanish, the rest remaining arbitrary, there will still be the same number of nodes, and the period of the vibration will still be $\frac{\tau}{i}$; but the form of the waves may be any whatever, as in Art. **93**.

If C_1 is not zero, there are no actual nodes (except the fixed ends), and the first component of the vibration consists in an oscillation of which the period is τ and the wave-length is twice the length of the string.

103. When a string, vibrating without nodes, produces an audible note, the lowest component tone heard is in general that of which the pitch corresponds to the period τ of the whole vibration, and this is called the fundamental tone of the string. But other tones, belonging to the harmonic scale of this fundamental tone, are in general heard also. (If, however, the period τ is so long that the fundamental tone does not fall within the limits of audibility, the lowest tone heard will of course be one of the higher components.)

If there is *one* node, the period becomes $\frac{\tau}{2}$, and the lowest tone heard is the *octave* of the fundamental tone.

And in general, if there are $i-1$ nodes, the period is $\frac{\tau}{i}$, and the lowest tone heard is the $(i-1)^{\text{th}}$ harmonic of the fundamental tone.

But in each case higher harmonic components are in general heard, so that the sound is a compound *note*.

The notes produced by a string vibrating with one or more nodes, are called by musicians the *harmonics* of the string.

104. Now, when the string vibrates without nodes, so as to produce what is called its fundamental note, the series of harmonic component tones is in general *complete* so far as it can be traced by the ear; and a practised ear, properly assisted (see below, Art. **115**), can easily distinguish ten or more. But we are able, as we shall see presently, to make a string vibrate in such a manner that for any proposed value of i all the coefficients C_i, C_{2i}, C_{3i}, &c. in the series (13) shall vanish; so that the component harmonic vibrations of which the periods are $\frac{\tau}{i}$, $\frac{\tau}{2i}$, &c. are extinguished. And it is found, as a fact, that when this is done, *the corresponding tones become either quite or nearly inaudible.*

105. From the facts stated in the last two Articles it is obvious to infer that each component tone actually heard, is produced exclusively, or at least mainly, by the corresponding component harmonic vibration of the string[1].

But to appreciate the force of this conclusion, we must consider the phænomena more precisely.

106. The vibrations in the ear which ultimately produce the sense of sound, are very remotely derived from those of the string. In the first place, the sound-waves in the air are excited in a very slight degree by the string itself. This may be shewn by stretching a violin or pianoforte string between two very firmly fixed supports—for instance, iron pegs in a wall—when it will be found impossible to make it yield a note of any considerable strength. In all actual stringed instruments, therefore, the supports of the string are so arranged as to communicate a state of forced vibration to a considerable surface of wood. Then this vibrating surface originates waves in the air; and these, being propagated to the tympanic membrane of the ear, put that membrane itself into a state of forced vibration, which is further communicated, by means of the linkwork of small bones mentioned in Art. **4**, to the membrane of the oval window; and finally, from that, through the fluid of the labyrinth, to those parts of which the vibrations ultimately affect the auditory nerve.

Evidently, therefore, it is essential to inquire, by what law the form and period of vibrations excited at any part of a material system by given vibrations maintained at any other part, are connected with the form and period of the latter.

[1] For a discussion of the question, What sort of vibrations produce simple tones? see the Memoirs of Ohm and Seebeck, in Poggendorf, vol. lix. p. 497; lx. 449; lxii. 1.

107. The answer to this question is contained in a statement of the *law of forced oscillations*. (See Appendix to this chapter.)

If a material system, acted on by a *conservative system* of forces[1], be very slightly disturbed from a configuration of stable equilibrium, and then left to itself after having had very small velocities (or none) impressed upon any of its particles, it will continue for ever to execute small oscillations; that is, every particle (except such as may remain at rest) will describe a path in which it will always be very near to the position which it had in the condition of equilibrium. The motion may or may not be a *vibration*, in the proper sense. That is, the system may, or may not, pass again and again at equal intervals of time through the same configuration.

108. If, however, the displacements and velocities are always so small that their squares and products may be treated as insensible, then the motion is sensibly either a true vibration, or else is compounded of vibrations each of which might subsist by itself, but of which the periods are in general incommensurable; so that by their superposition they produce a non-periodic motion, or more properly, a vibration of infinitely long period.

We may call these component vibrations the *natural* vibrations of the system, to distinguish them from the forced vibrations which are now to be considered.

109. In addition to the suppositions just made, let us now further suppose, either that certain points of the system are subjected to small obligatory periodic motions, or else to the action of small periodic forces; that is, forces of which the intensity is expressed by the product of two factors, one of which may be either constant or dependent on configurations, and the other is a periodic function of the time.

In either case, provided that all the displacements and velocities continue to be of the order of magnitude above supposed, the whole motion is compounded of two sets of vibrations:

[1] That is, a system in which the mutual action between any two particles is independent of the velocities of those and of all other particles. (Thomson and Tait, § 271.)

one which as before may be called *natural*, of which the periods are independent of the imposed motions or forces, and which might be entirely extinguished by a proper choice of the disposable initial displacements and velocities; and another set which are *forced* by the imposed motions or forces, and which are permanent, and in no way dependent on initial circumstances.

And it can be shewn that *no forced vibration can have any harmonic component of a period which does not exist amongst the periods of the harmonic components of the imposed motions or forces.*

110. We have so far supposed the original system of forces to be *conservative*, so that the natural vibrations, if once begun, would continue for ever. But in all actual cases there are *resistances* of various kinds, which sooner or later extinguish the natural vibrations. There is thus an apparent destruction of energy; but the energy is not really destroyed, but changed into other forms, which can in general be assigned, such as heat, &c.

When, however, we only wish to take account of the energy of the system in the ordinary mechanical sense, we have to introduce these resistances under the fictitious form of forces of the non-conservative class, that is, forces which are not independent of velocities; and on a particular hypothesis, which there is reason to believe gives sensibly correct results for small velocities, namely, that the motions of the particles are resisted by forces directly proportional to their velocities[1], no additional difficulty arises in the mathematical treatment of the problem, but the result is modified in the following manner.

[1] It must be understood that the word 'resistance' is here used to denote any cause which tends to extinguish that particular kind of motion which constitutes the vibration considered. In the case of a vibrating string, for instance, the resistance of the air is one such cause; another is the communication of motion from the ends of the string to the bodies which support its tension; and a third is probably the conversion of part of its energy into heat. The hypothesis is, that the combined effect of these causes may be represented by assuming a retarding force to act on each particle, directly proportional to its velocity; and it is at any rate certain that results calculated on this hypothesis agree in general much better with experience than those obtained by neglecting resistances altogether. Any other law of resistance would introduce insuperable difficulties into the mathematical treatment of most cases.

The *periods* of the natural vibrations are altered (in general slightly), without ceasing to be constant; but their amplitudes diminish rapidly, so that the system is soon brought sensibly to rest, if there are no obligatory motions or periodic forces.

If, however, there are, as we have above supposed, small periodic forces or obligatory periodic motions, the system soon assumes a permanent condition of motion, consisting of vibrations (which we will call *forced*, as before) of which the periods are connected with those of the imposed motions or forces according to the law already stated; and there is no trace of the natural vibrations except this: that the *amplitudes and phases* of the forced vibrations depend upon the relative magnitudes of their periods and those which the natural vibrations would have if they existed.

111. If any harmonic component of one of the imposed motions or forces have a period nearly equal to that of a harmonic component of any one of the natural vibrations, then there will be a corresponding forced vibration with a large amplitude. On the supposition of no resistances, and of absolute equality of periods, the amplitude in question would go on increasing with the time, so as soon to violate the supposition of small motions. But in all actual cases the effect of the resistances is to limit the increase of the amplitude to a definite magnitude, which, however, may still be larger than is consistent with the supposition referred to.

112. In fact, in a great number of cases, the supposition that the squares and products of displacements and velocities may be neglected leads to results which agree well, *as a first approximation*, with experiment, but fail to explain phænomena of a delicate but still perceptible kind, which can be accounted for by a second approximation.

The result of this second approximation shews that there are in general forced vibrations of which the amplitudes are small magnitudes of the second order, and of which the harmonic components have periods which are either the halves of the periods of harmonic components of the imposed motions or forces, or are such that the numbers of vibrations in a given time are the sums or differences of the numbers of vibrations of the latter taken two and two.

113. We shall shew afterwards how this result accounts for the remarkable phænomena of so-called 'combination-tones.' At present we see that, so far as the first approximation goes, we are entitled to assume that the vibrations which ultimately affect the auditory nerve have no harmonic components differing in period from those of the vibrations of the body from which the sound originates. And when we compare this theoretical conclusion with the observed fact that the extinction of any harmonic component of the vibration of a string, extinguishes (very nearly, if not entirely) the sensation of the corresponding tone, the inference appears unavoidable that the sensation of simple tone is produced by simple harmonic vibration.

114. We will now describe some simple experiments which exhibit the accordance of the theory of vibrating strings with facts.

In the first place, the isochronism of small vibrations (that is, the independence of their periods on their amplitudes) is shewn by the familiar fact that a given string produces a note of sensibly the same pitch, whether it be made to sound loudly or softly, provided the variations of loudness do not much exceed the limits usually allowed in music.

The point next in importance is the verification of the compound character of the note usually produced.

The following is an easy method of making some of the principal harmonic component tones sensible to an unpractised ear; or, rather, of making the ear conscious that it hears them. Strike any note of a pianoforte rather strongly, say c, and hold the key down so that the vibrations may not be stopped by the damper. (The two or three strings belonging to the note should be tuned well in unison.) Immediately afterwards strike very gently any note belonging to the harmonic scale of c, holding the key also down. Then, if the attention be fixed upon the sound of this latter note as it dies away, it will be heard to remain as a component of the note first struck;

and so distinctly, that it will often appear quite surprising that what is now a conspicuous phænomenon should have entirely escaped observation before attention was thus directed to it[1].

In this way eight or ten harmonic component tones may generally be distinguished. An ear which has been musically trained will soon acquire great facility in tracing these harmonic tones, up to a certain number varying with circumstances, without any assistance. But, in order to distinguish the higher and fainter ones, it is necessary to put the ear in communication with *resonators*, the action of which may be here briefly explained.

115. They are usually made of glass or brass, in the shape of nearly spherical bottles. The neck of the bottle is short, and so formed that by coating it with sealing-wax it may be made to fit closely into the outer part of the meatus of the ear. There is another orifice, opposite to the neck. When such a resonator is applied to the ear, it forms, with the meatus as far as the tympanic membrane, a cavity with one opening; and the air in such a cavity is capable of vibrating with a determinate period, which depends on the form and size of the cavity and of the opening.

Suppose now that there is an external vibrating body in the neighbourhood, the air in the cavity will be put into a state of *forced vibration*, of which the component periods will be those of the harmonic components of the vibrations of the external body, but of which the amplitudes will *in general* be inconsiderable. But if the period of any one of these harmonic components coincide exactly, or very nearly, with that of the natural vibration of the cavity, the amplitudes of the forced vibrations will be large (Art. **111**), and the ear will hear that particular component with great distinctness, and indeed often with unpleasant loudness. In order to obtain this effect in the highest degree, the other ear should be closed.

116. We shall have to refer afterwards to the general principles of resonance, and to the use of these resonators in particular.

[1] Helmholtz, p. 86.

Imperfect substitutes for them may be made of paste-board, and the following is an easy way of roughly illustrating their action. If a stiff paste-board tube, of about $1\frac{1}{4}$ inch in diameter, and of any length, from three or four inches upwards, be pressed with one end closely upon the ear, the tube and ear together form a cavity open at one end; and the note corresponding to the natural vibrations of this cavity is easily ascertained by tapping the outside of the tube with the ends of the finger-nails or with a pencil. The sound of the taps is not a mere *noise*, but has a determinate pitch, which, however, an unpractised ear is liable to estimate an *octave too low*. If, now, the corresponding note be struck on a pianoforte, and the coincidence of pitch be nearly exact, the effect of the tube in strengthening the fundamental tone of the string is very conspicuous, and may be made still more striking by alternately removing and replacing the tube. And if any other note be struck of which one of the stronger harmonic components has the pitch corresponding to the natural vibrations of the cavity, the strengthening of that component may be made more or less strikingly sensible in the same way.

By tilting the tube, so that it ceases to touch the ear all round, the pitch of the natural vibrations is *raised*, and can therefore be brought into coincidence with that of a higher tone. In this manner, by different angles of tilting, the same tube may be made to strengthen several harmonic components of the same note; and so, by tilting it backwards and forwards between complete contact and a considerable opening, this series of tones may be heard upwards and downwards several times before the vibrations of the string once struck cease to produce audible sound. This experiment requires a little care and practice, but when successful is very striking.

117. The vibration of a string with nodes is easily shewn on any instrument of the violin species, or on a horizontal pianoforte. In the latter case, if a key be struck and held down, while a finger is lightly applied at a nodal point, the string will sound the corresponding harmonic instead of its fundamental note. Or the finger may be applied *after* the key is struck, in which case the fundamental tone, which is heard at first, is extinguished, and the harmonic remains and is heard alone. Thus, if the fundamental note be c, and the finger be applied at one-third of the distance from either end of the string, the harmonic note g′ will be heard. In this case there are two nodes, and the existence of that one which is not touched by the finger may be shewn by placing on the string a small bent strip of paper: if it be placed at any point except the node, it will be shaken when the key is struck; but if at the node, it will remain undisturbed.

The harmonic notes of a harp string may be produced in the same way, and the first of them (or octave) is sometimes used by harp players.

The production of harmonic notes on the violin or violoncello, by touching the string lightly with the finger at the place of a node, is familiar to players on those instruments.

It is obviously essential to the production of harmonics that the point at which force is applied to make the string vibrate (whether by a hammer, the finger, or a bow) should *not* be a node.

118. It was stated above (Art. **104**) that a string may be made to vibrate in such a manner that any proposed harmonic component vibration, with all those whose periods are aliquot parts of the period of that one, shall be extinguished.

This follows from the formula (2) of Art. **99**; for if in this formula we suppose $a = \frac{m}{n} l$ ($\frac{m}{n}$ being a proper fraction in lowest terms), that is, if we suppose that the string is made to vibrate by being plucked at any one of the points which divide it into n equal parts, all those terms in the series vanish for which i is a multiple of n, and therefore the component vibrations of which the periods are $\frac{\tau}{n}$, $\frac{\tau}{2n}$, $\frac{\tau}{3n}$, &c. do not exist.

It was also stated that when this is done, the corresponding series of harmonic tones becomes nearly or quite inaudible[1].

[1] This fact was discovered by Dr. T. Young. See his 'Experiments and Inquiries respecting Sound and Light,' *Phil. Trans.* for 1800, p. 138.

119. To shew this experimentally, it is only necessary to pass the point of the finger very lightly across a pianoforte or violin string. This should be first done at a point *not* coinciding with a node in some proposed division, for instance, at a point *not* dividing the string into three equal parts, and the attention directed to the corresponding harmonic (in this case the third component, or 'twelfth' above the fundamental tone), so that it may be heard distinctly. Then, if the finger be passed across a node, the absence of the same harmonic will be unmistakeable.

The following is an easy and striking way of making this experiment. Pluck the string alternately, with a finger of each hand, at its middle point and at a point one-third of its length from either end. These points must be taken very exactly, and the fingers passed lightly across the string. Then along with the fundamental tone will be heard very distinctly the *twelfth* when the string is plucked at its *middle point*, and the *octave* when it is plucked at the other point. Thus, if the third[1] or d′ string of a violin be used, the tones a″, d″, will be heard alternately.

[1] The fourth string, which is covered with wire, does not answer so well. The wire covering appears to have the effect of immediately re-establishing the component vibrations which at the first instant were extinguished by the pluck.

APPENDIX TO CHAPTER VI.

ON FORCED OSCILLATIONS.

(The following is to be taken only as a slight sketch of the subject, in which many points of interest are omitted. For more complete details on that part which relates to *natural* oscillations, see Thomson and Tait, § 343. The process here given only differs from that employed by those authors in modifications of detail, and in the extension to the case in which the system is subjected to obligatory motions.)

If x, y, z be the coordinates of any point of a material system referred to fixed rectangular axes, the differential equations which define the motion of the system under the action of given external forces are to be derived from the formula

$$\Sigma m (x'' \delta x + y'' \delta y + z'' \delta z) = \Sigma (X \delta x + Y \delta y + Z \delta z), \qquad (1)$$

where accents signify total differentiation with respect to t, and the summation extends, on the left hand to every particle m, and on the right hand to every point at which an external force is applied. X, Y, Z are the components of the force applied at the point x, y, z.

This formula expresses the proposition (known as D'Alembert's theorem) that the system is at every instant in a configuration of equilibrium with respect to the applied forces and the resistances to acceleration arising from the inertia of the particles. When the system is not entirely free, the possible motions of the particles are limited by equations of condition; and δx, δy, &c. in (1) represent any arbitrary infinitesimal alterations which could, at the time t, be made in the coordinates x, y, &c. without violating those equations. In other words, the equation (1) must at every instant subsist for values of δx, &c. corresponding to any arbitrary infinitesimal displacement which the system could at that instant undergo without violating the conditions which limit the freedom of its motion.

When the equations of condition contain the time t explicitly, the expression 'configuration of equilibrium' at any proposed time is to be understood as meaning that which would be a configuration of equilibrium, if t, in the equations of condition, became constant, with the value which it has at the instant in question. (Thus, if a particle be constrained to move on a surface which is continually changing its form, it is in a configuration of equilibrium at any instant if the force applied to it is in the direction of a normal to the surface at that instant.)

We may call the equilibrium *relative* when the equations of condition contain t explicitly, and *absolute* when they do not.

Our present object is to consider the case in which the equations of condition contain t explicitly, only because given points of the system are subject to *obligatory motions*, so that the coordinates of those points are given functions of t. This, it should be observed, is only a generalization of the common case in which given points of the system are *fixed*. In either case the terms referring to those points disappear from both sides of the formula (1), because, whether a point is fixed, or subject to an obligatory motion, it could not, at any proposed time, receive any displacement without violating the condition imposed upon it; hence $\delta x = 0$, &c. for all such points.

Suppose, now, we refer the system to a set of *independent* coordinates $\xi_1, \xi_2, \ldots$ of any kind, that is, quantities of which the values at any time would determine the configuration of the system at that time, but could be all assumed arbitrarily without violating the equations of condition.

The transformation to the Lagrangian form[1] of the differential equations is in no way affected by the suppositions now made, so that we shall have as many equations of the second order as there are independent coordinates, viz.:

$$\left(\frac{dT}{d\xi_1'}\right)' - \frac{dT}{d\xi_1} = \Xi_1, \quad \text{\&c.} \tag{2}$$

on which equations, however, the following observations are to be made.

T is the expression, in terms of the new variables ξ_1, &c. of

$$\tfrac{1}{2}\Sigma m\,(x'^2 + y'^2 + z'^2);$$

the summation extending to all coordinates which appear on the left-hand side of (1). The coordinates of those points

[1] See Price, *Inf. Cal.*, vol. iv., § 302.

which are subject to obligatory motions therefore do not appear in T as expressed in terms of x, y, z, &c.

Let us denote by x, y, z, &c. the original coordinates of those points which *are* subject to obligatory motions. Then the relations between the old and new variables will enable us to express each of the other coordinates x, y, z, &c. as a function of $\xi_1, \xi_2, \ldots$ and x, y, z, &c.; suppose, for instance,

$$x = \Psi(\xi_1, \xi_2, \ldots \mathrm{x}, \mathrm{y}, \mathrm{z}, \ldots);$$

then

$$x' = \frac{dx}{d\xi_1}\xi'_1 + \frac{dx}{d\xi_2}\xi'_2 + \ldots + \frac{dx}{d\mathrm{x}}\mathrm{x}' + \ldots \quad \text{\&c.};$$

and when these values are substituted in the original expression for T, the result is evidently a function which is homogeneous, and of the second degree, with respect to $\xi'_1, \xi'_2 + \ldots \mathrm{x}', \mathrm{y}', \ldots$ Now $\mathrm{x}, \mathrm{y}, \ldots \mathrm{x}', \mathrm{y}', \ldots$ are given functions of t; and when their values are introduced T is no longer homogeneous with respect to $\xi'_1, \xi'_2, \ldots$ and also contains t explicitly.

(This new value of T is merely the kinetic energy (or half *vis-viva*) of the whole system expressed in terms of the independent coordinates.)

For the present, however, we will suppose T to be expressed in terms of $\xi'_1, \ldots, \mathrm{x}', \ldots$, so that we may assume

$$\begin{aligned} T = \tfrac{1}{2}P\xi'^2_1 + \tfrac{1}{2}Q\xi'^2_2 + \ldots + \tfrac{1}{2}R\mathrm{x}'^2 + \ldots \\ + S\xi'_1\xi'_2 + \ldots + T\xi'_1\mathrm{x}' + \ldots + U\mathrm{x}'\mathrm{y}' + \ldots \end{aligned} \qquad (3)$$

where the coefficients P, Q, &c. are given functions which may contain all the coordinates, but not their differential coefficients ξ'_1, &c.

The functions $\Xi_1, \Xi_2, \ldots$ in equations (2) are to be found by means of the equation

$$X\delta x + Y\delta y + Z\delta z + \ldots = \Xi_1\delta\xi_1 + \Xi_2\delta\xi_2 + \ldots, \qquad (4)$$

which gives

$$\Xi_1 = X\frac{dx}{d\xi_1} + Y\frac{dy}{d\xi_1} + Z\frac{dz}{d\xi_1} + \ldots, \quad \text{\&c.};$$

so that $X, Y, \ldots$ being given functions of the original coordinates, Ξ_1, &c. are known functions of the new coordinates, containing also in general x, y, &c.

We have now to introduce the supposition that the obligatory motions consist of small vibrations. Let us assume, for instance, that the point (x, y, z) is subject to a vibration, so that

$$\mathrm{x} = \mathrm{x}_0 + au, \qquad \mathrm{y} = \mathrm{y}_0 + av, \qquad \mathrm{z} = \mathrm{z}_0 + aw;$$

where $\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0$ are constants, u, v, w are given periodic functions of t, and a is a constant which may be considered a small quantity of the first order. We may call (x_0, y_0, z_0) the *mean* position of the point (x, y, z); (the point (x, y, z) would become fixed at (x_0, y_0, z_0) if a were $= 0$).

Next we will suppose that the system is always *nearly* in a configuration of absolute stable equilibrium; that is, that the values of the coordinates $\xi_1, \xi_2, \ldots$ only differ by quantities of the first order from values (which we will denote by (ξ_1) &c.) which would belong to a configuration of stable equilibrium if a were $= 0$.

The excursions of the particles being of the first order, we make the further assumption (which must be justified[1] *à posteriori*) that the differential coefficients ξ'_1, &c. are also of the first order. Hence T is of the second order. But $\dfrac{dT}{d\xi'_1}$, &c. are of the first order, while $\dfrac{dT}{d\xi_1}$, &c. are of the second (since ξ_1, &c. only occur in the coefficients P, &c.). Hence, if we retain only terms of the first order, we must reject the terms $\dfrac{dT}{d\xi_1}$, &c. in the equations (2).

Moreover, it is evident that in the coefficients P, Q, &c. we may put x_0 for x, &c. and the equilibrium values for ξ_1, &c., so that those coefficients will receive constant values. Hence we may take

$$\begin{aligned} T = \tfrac{1}{2}[1, 1]\,\xi'^2_1 + \tfrac{1}{2}[2, 2]\,\xi'^2_2 + \ldots + [1, 2]\,\xi'_1\xi'_2 + \\ + a_1\xi'_1 x' + b_1\xi'_1 y' + c_1\xi'_1 z' \\ + a_2\xi'_2 x' + \ldots \end{aligned}$$

+ terms in x', y', ... only, which will disappear in forming equations (2).

Hence the left-hand member of the first of equations (2) becomes

$$[1, 1]\,\xi''_1 + [1, 2]\,\xi''_2 + [1, 3]\,\xi''_3 + \ldots + a_1 x'' + b_1 y'' + c_1 z'',$$

and of the second equation—

$$[2, 1]\,\xi''_1 + [2, 2]\,\xi''_2 + [2, 3]\,\xi''_3 + \ldots + a_2 x'' + b_2 y'' + c_2 z'',$$

and so on. In these equations the symbols $[1, 1]$, &c. as well as a_1, &c. merely denote given constants, and $[1, 2] = [2, 1]$, &c.

[1] The justification is this: The motions of the particles consist of *isochronous* vibrations, from which it follows that the velocities may be diminished indefinitely by diminishing the amplitudes. The supposition of small velocities therefore merely implies a certain limit to the allowable magnitude of the amplitudes.

With respect to the right-hand members of (2) we observe that, putting as before (ξ_1), &c. for the equilibrium values of ξ, &c. we shall have, by Taylor's theorem, as far as terms of the first order,

$$\Xi_1 = (\Xi_1) + (\xi_1 - (\xi_1))\left(\frac{d\Xi_1}{d\xi_1}\right) + (\xi_2 - (\xi_2))\left(\frac{d\Xi_1}{d\xi_2}\right) + \dots$$
$$+ a\left(u\left(\frac{d\Xi_1}{dx}\right) + v\left(\frac{d\Xi_1}{dy}\right) + w\left(\frac{d\Xi_1}{dz}\right)\right),$$

where brackets signify values corresponding to equilibrium values of the coordinates.

Now, when the variables have equilibrium values

$$\Sigma\,(X\,\delta x + Y\delta y + Z\,\delta z) = 0,$$

and therefore

$$(\Xi_1)\,\delta\,\xi_1 + (\Xi_2)\,\delta\,\xi_2 + \dots = 0.$$

But since $\delta\,\xi_1$, &c. are independent and arbitrary, this implies $(\Xi_1) = 0$, $(\Xi_2) = 0$, &c.

Further, it is evident that we may assume the zeros of the coordinates ξ_1, &c. in such a manner that their equilibrium values (ξ_1), &c. shall $= 0$. Hence, if we denote the constants $\left(\frac{d\Xi_1}{d\xi_1}\right)$, &c. by letters A, B, &c. the first of equations (2) will become

$$[1, 1]\,\xi''_1 + [1, 2]\,\xi''_2 + \dots + a_1 x'' + b_1 y'' + c_1 y''$$
$$= A_1\,\xi_1 + A_2\,\xi_2 + \dots + a\,(fu + gv + hw).$$

Now $x = x_0 + a\,u$, &c. so that $x'' = a\,u''$, &c. We have also supposed that u, v, w are periodic functions of t, and therefore each of them may be expressed as a series of simple harmonic terms by means of Fourier's theorem. Suppose $\sin(nt + \beta)$ occurs in the value of x, then it will also occur in x″; and we now see that, putting D for $\frac{d}{dt}$, we may write the above equation in the form

$$([1, 1]\,D^2 - A_1)\,\xi_1 + ([1, 2]\,D^2 - A_2)\,\xi_2 + \dots = k \sin(nt + \beta) + \dots,$$

the right-hand side consisting entirely of harmonic terms of different periods, multiplied by small coefficients. The second equation will be of the form

$$([2, 1]\,D^2 - B_1)\,\xi_1 + ([2, 2]\,D^2 - B_2)\,\xi_2 + \dots = l \sin(nt + \gamma) + \dots,$$

and so on. Thus we shall have a set of simultaneous differential equations of the second order, as many in number as there are independent coordinates.

We will now, however, introduce the further supposition of small resistances varying directly as the velocities of the particles. This does not add any difficulty to the integration of the equations, and leads to results more in accordance with experience.

This supposition is equivalent to writing $X - \epsilon \frac{dx}{dt}$, &c. instead of X, &c. in some or all of the terms of the original formula, ϵ being a small constant; and it is evident that the effect of this will be, when terms of the second order are neglected, to add to Ξ_1, &c. linear functions of ξ'_1, ξ'_2, &c. with constant coefficients; and transposing these terms to the left-hand side of the differential equations, we shall have (using a_1, &c. in a new sense),—

$$([1, 1]D^2 + a_1 D - A_1)\,\xi_1 + ([1, 2]D^2 + a_2 D - A_2)\,\xi_2 + \ldots = k \sin(nt + \beta) + \ldots$$

$$([2, 1]D^2 + b_1 D - B_1)\,\xi_1 + ([2, 2]D^2 + b_2 D - B_2)\,\xi_2 + \ldots = l \sin(nt + \gamma) + \ldots$$

$$\cdots\cdots\cdots$$

which are the differential equations of the problem.

This system of equations may be treated in several different ways, of which the following appears most convenient for our present purpose.

Adopting abbreviations for the operative symbols, we may write the equations thus:

$$[a\,a]\,\xi_1 + [a\,b]\,\xi_2 + \ldots = k \sin(nt + \beta) + \ldots$$
$$[b\,a]\,\xi_1 + [b\,b]\,\xi_2 + \ldots = l \sin(nt + \gamma) + \ldots$$
$$\cdots\cdots$$

(where it is to be observed that $[a\,b]$ is not the same as $[b\,a]$).

Let ∇ be the determinant

$$\begin{vmatrix} [a\,a], & [a\,b], & [a\,c]\ldots \\ [b\,a], & [b\,b], & [b\,c]\ldots \\ [c\,a], & [c\,b], & [c\,c]\ldots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

Now if we operate upon the differential equations respectively with the minor determinants

$$\frac{d\nabla}{d[a\,a]} \qquad \frac{d\nabla}{d[b\,a]} \qquad \frac{d\nabla}{d[c\,a]} \qquad \ldots$$

just as in a system of algebraic linear equations, ξ_2, ξ_3, &c. will be eliminated; and, observing the effect of differentiation on the periodic terms in the right-hand members, we see that the result will be

$$\nabla\ \xi_1 = K \sin(nt + L) + \ldots \qquad (5)$$

and in like manner

$$\nabla\ \xi_2 = K' \sin(nt + L') + \ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

K, K' L, &c. being known constants, and K, K', ... small.

To integrate these equations we first suppose their right-hand members to be absent, so that all the variables ξ_1, ξ_2, ... satisfy the same differential equation, which we may write

$$\nabla\ u = 0,$$

and which is of the order $2m$, if m be the number of the independent coordinates ξ_1, &c.

If we put for a moment $\nabla = f(D)$, and call a_1, a_2, a_3, ... a_{2m} the roots of the equation $f(x) = 0$, then we know that the general solution of $\nabla u = 0$ is

$$u = C_1 \epsilon^{a_1 t} + C_2 \epsilon^{a_2 t} + \ldots.$$

C_1, C_2, ... C_{2m} being arbitrary constants. Hence the values of the variables ξ_1, ξ_2, ... are of this form, differing only in the values of the constants C_1, &c.; but since the complete solution of the system can only contain $2m$ arbitrary constants, those in the values of ξ_2, &c. must be expressible as functions of those in ξ_1. The relations necessary for this purpose would have to be ascertained by substituting the expressions for ξ_1, &c. in the original system of equations of the second order. We have no occasion, however, to perform this operation actually.

Considering now the value of any coordinate, say

$$\xi_1 = C_1 \epsilon^{a_1 t} + \ldots,$$

we see that the motion cannot consist of permanently *small* oscillations, unless the roots of the equation $f(x) = 0$ consist of imaginary pairs, and the real part of each pair be negative, so that the value of ξ_1 may be put in the form

$$\xi_1 = C_1 \epsilon^{-a_1 t} \sin(m_1 t + \beta_1) + C_2 \epsilon^{-a_2 t} \sin(m_2 t + \beta_2) + \ldots \qquad (6)$$

(where a_1 &c. are used with a new meaning). The factors $\epsilon^{-a_1 t}$ &c. are introduced solely by the resistances, as is evident if we observe that the differential equations would only contain even powers of D if there were no resistances.

The above values of ξ_1, &c. determine the natural vibrations of the system, such as could exist if the obligatory vibrations were suppressed by fixing the points subject to them; and they are compounded of harmonic vibrations, each of which could subsist alone. We see also that the effect of the resistances would be gradually to diminish the amplitudes of the vibrations so that the motion would be ultimately extinguished.

We have thus obtained the complete solution of the system of equations (5) &c., on the supposition that K, K', &c. are all 0. And the solution of those equations in their actual form may now be found as follows:

Assume, for the complete value of ξ_1, the terms in the expression (6)

$$+ A \sin(nt + L) + B \cos(nt + L) + \ldots$$

where $A, B \ldots$ are constants to be determined.

(A similar pair of terms is to be assumed for every term on the right-hand side of (5); but as the following process would merely have to be repeated for each pair, we shall attend only to the first.)

Substituting this value of ξ_1 in (5), and observing that the operation ∇ destroys the part (6), we obtain the condition

$$A \nabla \sin(nt+L) + B \nabla \cos(nt+L) = K \sin(nt+L) \quad (7)$$

for the determination of A and B.

Now the operation ∇, consisting partly of even and partly of odd powers of D $\left(\text{or } \frac{d}{dt}\right)$, may be put in the form

$$\nabla = \phi(D^2) + D\chi(D^2),$$

where the second term is introduced entirely by the resistances, and if they are considered as small quantities of the first order the terms depending on them in $\phi(D^2)$ will be of higher orders, so that $\phi(D^2)$ may be considered as what ∇ would be reduced to if there were no resistances. The result of the operations on the left-hand side of (7) is easily seen to be

$$(A\phi(-n^2) - Bn\chi(-n^2)) \sin(nt+L)$$
$$+ (An\chi(-n^2) + B\phi(-n^2)) \cos(nt+L),$$

and this is to be identical with $K \sin(nt+L)$; hence we must have

$$A\phi(-n^2) - Bn\chi(-n^2) = K,$$
$$An\chi(-n^2) + B\phi(-n^2) = 0;$$

from which we find

$$\frac{A}{\phi(-n^2)} = \frac{-B}{n\chi(-n^2)} = \frac{K}{\{\phi(-n^2)\}^2 + n^2\{\chi(-n^2)\}^2};$$

and when the values of A and B thus given are introduced in the expression assumed for ξ_1, it becomes (after an obvious reduction)

$$\xi_1 = C_1 \epsilon^{-\alpha_1 t} \sin (m_1 t + \beta_1) + \dots$$
$$+ \frac{K}{\{(\phi(-n^2))^2 + n^2 (\chi(-n^2))^2\}^{\frac{1}{2}}} \sin (nt + L + \theta) + \dots, \quad (8)$$

θ being a constant of which we need not write the actual value.

Thus we see that the effect of every harmonic term in the obligatory vibrations is in general to add to the value of each coordinate a term with the same period; and these added terms represent the *forced vibrations* of the system. The other terms, which are the same as if $K = 0$, &c. give the *natural vibrations*, but these in general soon become insensible through the diminution of the factors $\epsilon^{-\alpha_1 t}$, &c. introduced by the resistances, while the forced vibrations are permanent.

It is important to observe that it follows from the nature of the whole process that no periods will be introduced in the forced vibrations which do not exist in the harmonic components of the obligatory vibrations, so long as terms of the first order only are considered, because no periodic terms are ever *multiplied together*.

With respect to the *amplitudes* of the forced vibrations, we see that those will be large of which the periods are such that the denominators of the coefficients in (8) are small.

Now in the expression

$$(\phi(-n^2))^2 + n^2 (\chi(-n^2))^2,$$

the second term is introduced by the resistances, and is therefore *small*; hence the vibration, $\left(\text{of which the period is } \frac{2\pi}{n}\right)$, will have a large amplitude if n be such that $\phi(-n^2) = 0$; but the roots of the equation $\phi(x^2) = 0$ give the values of

$$\pm m_1 \sqrt{-1}, \text{ \&c.}$$

on the supposition that there are no resistances, so that on that supposition $\phi(-m_1^2) = 0$, &c.

Hence it follows that the amplitude of any component of the forced vibration will be large if its period coincide with that which would belong to a component of the natural vibrations if there were no resistances. And since the effect of the resistances in altering the periods of the natural vibrations is in general small, we may say in general terms that there will be forced vibrations of large amplitude if amongst the harmonic components of the obligatory vibrations there exist any with periods equal to those of natural vibrations.

Lastly, we see from (8) that though the periods of the forced vibrations at all parts of the system are the same as those of the obligatory vibrations which give rise to them, the phases are in general different.

The process to be used in the case of periodic forces is nearly identical with that which has been just explained, but somewhat simpler, and the results are exactly of the same kind. It is therefore omitted.

CHAPTER VII.

ON THE TRANSVERSE VIBRATIONS OF AN ELASTIC STRING. (*Dynamical Theory.*)

120. The rigorous differential equations which define the motion of an elastic string under given conditions, can in general be formed without difficulty, but cannot be integrated. It is only when the circumstances of the problem are such that certain quantities involved in the equations may be neglected without sensible error, that an integrable approximate form is obtained. For our present purpose it will not be necessary to form the rigorous equations; we shall introduce *ab initio* the conditions of the actual problem, and neglect the small quantities mentioned above, so as to obtain the approximate equations directly.

Let us then consider an elastic string, of which the ends are fixed at two points A, B, at a distance l from one another, greater than the natural length of the string.

In the condition of equilibrium the form of the string will be a straight line of length l, and it will have a tension, T, constant throughout its length, depending on the amount of the extension to which it has been subjected.

If now the string be slightly disturbed and then left to itself, its motion will be such that no particle will ever be much displaced from its position of equilibrium, and we make the following assumptions:—

(1) The original disturbance is such that the vibrations are sensibly transversal; that is, the projection of any particle on the line AB may be regarded as a fixed point.

(2) The inclination of every part of the string to the line AB is always an angle so small that the square of its sine or tangent may be neglected.

(3) The extension of any portion of the string due to its change of form may be neglected; or the ratio of the actual length to the length in equilibrio differs insensibly from unity. (This is evidently a consequence of (2).)

(4) The tension is sensibly constant, and may therefore be taken as always equal to T. (This is a consequence of (3).)

121. Now let the position of any element of the string be referred to rectangular axes having their origin at A, and the axis of x coinciding with AB.

It follows from (1) that, for a given element, x is to be regarded as constant, and from (2) that $dx = ds$. Consider now an element of which the length is ds, or dx, and let x, y, z be the coordinates of that end of it which is nearest to A, and $x+dx$, $y+dy$, $z+dz$ of the other end.

If no external forces are taken account of, the element is only acted on by the tensions at its extremities; and the components of these, in the directions of the coordinate axes, are

$$-T\frac{dx}{ds}, \quad -T\frac{dy}{ds}, \quad -T\frac{dz}{ds} \text{ at the end next } A,$$

$$T\frac{dx}{ds}+d\left(T\frac{dx}{ds}\right), \quad T\frac{dy}{ds}+d\left(T\frac{dy}{ds}\right), \quad T\frac{dz}{ds}+d\left(T\frac{dz}{ds}\right),$$

at the other end; so that the whole components are

$$d\left(T\frac{dx}{ds}\right), \quad d\left(T\frac{dy}{ds}\right), \quad d\left(T\frac{dz}{ds}\right).$$

But since T is constant and $ds = dx$, these become

$$0, \quad Td\left(\frac{dy}{dx}\right), \quad Td\left(\frac{dz}{dx}\right).$$

Hence if dm be the mass of the element, we have

$$dm \cdot \frac{d^2y}{dt^2} = Td\left(\frac{dy}{dx}\right), \quad dm \cdot \frac{d^2z}{dt^2} = Td\left(\frac{dz}{dx}\right). \qquad (1)$$

Let ρ be the longitudinal density of the string, or the mass of a unit of length in its actual state of extension. (The string being supposed uniform, ρ will be constant.) Then

$$dm = \rho\, ds = \rho\, dx,$$

and if we put $\frac{T}{\rho} = a^2$, and take dx constant, we obtain from (1) the equations

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2}, \qquad \frac{d^2z}{dt^2} = a^2 \frac{d^2z}{dx^2}. \tag{2}$$

The integrals of these equations will determine y and z as functions of the two independent variables x and t; that is, they will give the position, at the time t, of any proposed particle. (The value of x defines the particle, and the values of y, z determine its displacement from the position of equilibrium.)

122. Since the first of equations (2) does not contain z, and the second does not contain y, it follows that the motion of the projection of the string on the plane of xy is independent of that of its projection on the plane of yz; so that it will be sufficient to discuss one of these equations; and we will suppose for simplicity that the motion corresponding to the other does not exist. This is evidently equivalent to supposing that the displacements and velocities of the particles produced by the original disturbance were all in one plane, which we will take to be that of xy.

We have then the single equation

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2}, \tag{3}$$

of which the general solution is

$$y = f(x - at) + F(x + at). \tag{4}$$

This equation represents the transmission of two arbitrary forms along an unlimited line, with the same velocity a, (corresponding to the v of Art. **96**), in contrary directions; and we have already shewn (Arts. **88**, &c.) what must be the general character of these forms in order that the resultant curve may have *nodes*, so that a portion of the infinite string contained between any two nodes may be considered as a finite string fixed at its two ends. The mode of obtaining the same conclusions from equation (4) in a more analytical manner may be seen in treatises on mechanics. (See, for example, Poisson, *Traité de Mécanique*, t. ii. chap. viii. Price, *Inf. Cal.* vol. iv. § 281.)

It was also shewn how the most general solution of (3) in which the functions f and F satisfy the conditions imposed by the problem, may be expressed in the form

$$y = \sum_{i=1}^{i=\infty} \sin\frac{i\pi x}{l}\left(A_i \cos\frac{2i\pi t}{\tau} + B_i \sin\frac{2i\pi t}{\tau}\right), \qquad (5)$$

where $a\tau = 2l$.

123. The value of a^2 is $\dfrac{T}{\rho}$; let W be the weight of the string, then $W = g\rho l$, and $a^2 = \dfrac{Tgl}{W}$, and τ, the time of a vibration, is therefore $2\sqrt{\dfrac{Wl}{Tg}}$. For a string of given material and thickness, $W \propto l$, and therefore τ varies directly as the length and inversely as the square root of the tension. If the material only be given, W is proportional to the length and to the square of the thickness, so that τ varies as the length and the thickness directly, and the square root of the tension inversely.

If c be the length of string of which the weight would equal the tension T, then $T = g\rho c$, and therefore $\tau = \dfrac{2l}{\sqrt{gc}}$. Hence if g be expressed in the usual manner, the number of vibrations in a second is $\dfrac{1}{2l}\sqrt{gc}$.

124. The form (5) may be arrived at directly as follows:

The form of equation (3) suggests at once as a particular solution

$$y = \sin mx\,(A\cos amt + B\sin amt)$$
$$+ \cos mx\,(C\cos amt + D\sin amt),$$

in which, as long as no particular conditions are specified, all the constants m, A, B, C, D are arbitrary. But in order that this solution may represent the motion of the finite string, we must have, for all values of t, $y=0$ when $x=0$, and also when $x=l$. The first of these two conditions gives $C=0$, $D=0$; and the second gives

$$\sin ml = 0$$

whence $ml = i\pi$, i being any positive or negative integer. Thus we obtain

$$y = \sin\frac{i\pi x}{l}\left(A\cos\frac{i\pi at}{l} + B\sin\frac{i\pi at}{l}\right)$$

as a particular solution; and since the sum of any number of particular solutions is a solution of a linear differential equation, we obtain (putting $2a\tau = l$) the form (5) as a more general solution.

The coefficients A_i, B_i are all arbitrary unless the initial circumstances of the motion are given; but it has been shewn already (Art. **98**) that they can be so determined as to give the required initial values to y and $\frac{dy}{dt}$ for every part of the string. The form (5) therefore is the most general solution of the differential equation applicable to this simple case of the lateral vibration of a string. (Longitudinal vibrations will be considered in connection with those of a rod.)

125. The problem becomes little more complicated if we introduce the supposition of a retarding force acting at every point of the string, and proportional to the velocity [1]. The first of equations (1) then becomes

$$dm\,.\frac{d^2y}{dt^2} = Td\left(\frac{dy}{dx}\right) - c\,dx\,.\frac{dy}{dt},$$

c being a constant. And instead of the first of (2) we shall have

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} = a^2\frac{d^2y}{dx^2}, \tag{2'}$$

where $2k$ is put for $\frac{c}{\rho}$.

Multiplying (2′) by ϵ^{kt}, and using the theorem

$$\epsilon^{kt} f\left(\frac{d}{dt}\right) y = f\left(\frac{d}{dt} - k\right)\epsilon^{kt} y,$$

[1] See Art. **110**, *note*.

we obtain

$$\left\{\left(\frac{d}{dt}\right)^2 - k^2 - a^2\left(\frac{d}{dx}\right)^2\right\}\epsilon^{kt} y = 0.$$

Assuming as a solution of this equation

$$\epsilon^{kt} y = \sin(mx + a)\sin(pt + \beta),$$

we find the condition

$$-p^2 - k^2 + a^2 m^2 = 0,$$

$$\text{or}\quad p = (a^2 m^2 - k^2)^{\frac{1}{2}};\quad \text{hence}$$

$$y = \epsilon^{-kt}\sin(mx + a)\sin((a^2 m^2 - k^2)^{\frac{1}{2}} t + \beta) = 0$$

is a solution.

The ends of the string being fixed, we must have (for all values of t) $y = 0$ when $x = 0$, and when $x = l$; and therefore

$$\sin a = 0, \qquad \sin(ml + a) = 0.$$

It is easily seen that we lose no generality by taking $a = 0$; then $ml = i\pi$, or $m = \frac{i\pi}{l}$. Finally, therefore, the most general solution of (2′) appropriate to the problem is

$$y = \epsilon^{-kt}\sum_{i=1}^{i=\infty} A_i \sin\frac{i\pi x}{l}\sin\left(\left(\frac{i^2\pi^2 a^2}{l^2} - k^2\right)^{\frac{1}{2}} t + \beta_i\right); \quad (5')$$

in which A_i, β_i are arbitrary constants, to be determined as usual by initial circumstances.

From the above expression we see that the amplitude of the vibrations will progressively diminish, and ultimately become insensible, through the diminution of the factor ϵ^{-kt}.

Also that the period of vibration of the i^{th} tone is increased by the resistance in the ratio of 1 to $\left(1 - \frac{k^2 l^2}{i^2\pi^2 a^2}\right)^{\frac{1}{2}}$.

If the value of k were so great that, for any values of i, $k > \frac{i\pi a}{l}$, the value of y would contain non-periodic terms. But we shall not discuss this case, as it does not concern us practically.

126. We proceed now to some problems of a less simple kind, but of more or less practical interest.

Problem 1. *To find the motion of the string when a given point of it is subject to a given obligatory transverse vibration.*

We suppose, in the first instance, that there is no resistance.

Let b be the distance of the given point from the end A of the string; and suppose that at this point the value of y is to be $k \sin nt$. (We have no occasion to consider the external forces necessary to maintain this obligatory motion; we suppose them to be applied, whatever they may be.)

It is evident that the portions of string on the two sides of the given point may move independently of one another, except that the value of y must be the same for both when $x = b$.

Let us assume then that from $x = 0$ to $x = b$,

$$y = \sin mx \, (A \cos mat + B \sin mat).$$

This satisfies the differential equation, and also the condition that $y = 0$ when $x = 0$.

We have next to satisfy the condition that when $x = b$, $y = k \sin nt$. This gives

$$\sin mb \, (A \cos mat + B \sin mat) = k \sin nt,$$

which can only be true for all values of t on the supposition that $A = 0$, $ma = n$; hence

$$B = \frac{k}{\sin \frac{nb}{a}},$$

and the value of y becomes (for this part of the string)

$$y = \frac{k}{\sin \frac{nb}{a}} \,.\, \sin \frac{nx}{a} \,.\, \sin nt. \qquad (6)$$

This however is only a particular solution, since it contains no arbitrary constant. But it is evident that we may still satisfy the differential equation, without violating the prescribed condition, if we add the terms which would give the natural vibration of this part of the string, if the given point were fixed. Thus we get

$$y = \frac{k}{\sin \frac{nb}{a}} \sin \frac{nx}{a} \sin nt$$

$$+ \sum_{i=1}^{i=\infty} \sin \frac{i\pi x}{b} \left(A_i \cos \frac{ai\pi t}{b} + B_i \sin \frac{ai\pi t}{b}\right), \qquad (6')$$

and the series of arbitrary constants A_i, B_i, will enable us to satisfy the initial conditions relative to this part of the string, so that the above equation gives the general solution of the problem.

The motion of the other part may be found exactly in the same way, and will be given merely by writing $l-x$ and $l-b$ instead of x and b in (6′).

127. Attending for the present only to the first portion of the string, we see that the value of y in (6′) consists of two parts, the first of which depends only on the imposed obligatory motion, and determines the *forced* vibration of that portion. The other part is independent of the obligatory motion, and represents the *natural* vibration.

In any actual case the natural vibration is soon extinguished, because the string is constantly giving up some of its momentum to the air and to the bodies which support the tension of its ends. But the forced vibration will continue as long as the obligatory motion is sustained.

If we call τ the period of this forced vibration, so that $\tau = \frac{2\pi}{n}$, and neglect the natural vibration, the equation (6) may be written

$$y = \frac{k}{\sin \frac{2\pi b}{a\tau}} \cdot \sin \frac{2\pi x}{a\tau} \cdot \sin \frac{2\pi t}{\tau}, \tag{7}$$

in which it is to be remembered that k is the amplitude of the vibration imposed upon a point at the distance b from one end, so that b is the length of the portion of string now considered. For any given point at a distance x from the fixed end the amplitude of the forced vibration is therefore

$$\sin \frac{2\pi x}{a\tau} \cdot \frac{k}{\sin \frac{2\pi b}{a\tau}},$$

which expression becomes ∞ (except for $x = 0$) if $\sin \frac{2\pi b}{a\tau} = 0$. This cannot of course actually happen; in fact, the whole of the preceding reasoning fails if the excursions of any part of the string become large. All that we can really infer therefore is that when $\sin \frac{2\pi b}{a\tau}$ is 0, or very small, the amplitude of the forced vibrations will be much greater than under other conditions. Now $\sin \frac{2\pi b}{a\tau} = 0$ gives $\frac{2\pi b}{a\tau} = i\pi$, or $2b = ia\tau$.

Suppose λ the length of a portion of the string which would have τ for the period of its natural vibration; then $2\lambda = a\tau$ (Art. **94**), and therefore the above condition is equivalent to

$b = i\lambda$; that is,

The amplitude of the forced vibrations in this portion of the string becomes large when its length is any multiple of that which would vibrate naturally in the same period; or, which is the same thing, when the tone corresponding to the period of the forced vibration belongs to the harmonic scale of tones which would be given by the natural vibrations of this portion if both ends were fixed.

128. The points of minimum disturbance will be those at which $\sin\frac{2\pi x}{a\tau}$ vanishes; that is, at which $\sin\frac{\pi x}{\lambda} = 0$; or, on the supposition now made, $\sin\frac{i\pi x}{b} = 0$; that is, at points which divide the length b into i equal parts, so that there will be *quasi nodes* at these points. Similar conclusions may be deduced for the other portion of the string; and it is easily seen that if the obligatory vibration be imposed at any one of the points of division of the whole string into i equal parts, and if its period be $\frac{1}{i}$th of the period of the natural vibrations of the whole string, the forced vibrations will be large, and the tone will be the i^{th} of the harmonic scale of the string.

Thus we learn that in order to produce strong forced vibrations in a string, the obligatory vibration must be imposed at what would be a *node* in the case of natural vibrations of the same period; a conclusion which may appear strange. It might have been conjectured that a point of *greatest motion* ought to have been chosen. The explanation is simple, and may be left to the reader.

129. We have so far supposed, for simplicity, that the obligatory motion was a simple harmonic vibration. But if it were a compound vibration consisting of the superposition of any number of harmonic vibrations of different periods, phases, and amplitudes, it would be easily shewn that every one of these components would produce a corresponding term (analogous to (7)) in the expression for the forced vibrations of the string.

Thus, if the imposed vibration required that, when $x = b$,

$$y = \Sigma\left(A_i \cos\frac{2\pi t}{\tau_i} + B_i \sin\frac{2\pi t}{\tau_i}\right),$$

the forced vibration of the string, from $x = 0$ to $x = b$, would be expressed by the equation

$$y = \Sigma \frac{\sin \frac{2\pi x}{a\tau_i}}{\sin \frac{2\pi b}{a\tau_i}} \left(A_i \cos \frac{2\pi t}{\tau_i} + B_i \sin \frac{2\pi t}{\tau_i}\right).$$

In this case, if $\tau_i = \frac{2l}{ia}$: that is, if the period of every component of the imposed vibration is some aliquot part of the natural period of the whole string, then the above expression is not altered by changing x and b into $l-x$ and $l-b$; and therefore it holds good for the whole string.

130. These results may be approximately verified by experiment as follows: If a tuning-fork be struck, and the end of its stalk be then placed on the string of a pianoforte, violin, or violoncello, at any point, it may be considered approximately as imposing an obligatory vibration on that point. And it will be found that in general the only sound heard is the note of the tuning-fork, very weak. If, however, the point of application be such that the portion of string intercepted between it and either end could vibrate naturally so as to give, either as a fundamental tone or as a harmonic, one of the component tones of the tuning-fork, then that portion of the string is thrown into strong vibration, so as to give the corresponding tone very distinctly.

The fundamental tone of a tuning-fork is by far the strongest of its component tones. The higher components *proper*, are very high tones with incommensurable periods, which are hardly heard after a few seconds. But there is also a harmonic tone, an octave above the fundamental tone, which is weak, but persistent, and this, as well as the fundamental tone, may be produced from a string in the manner just described [2].

For instance, if a c'' tuning-fork be placed on the second string of a violin at the point where the finger would be placed in playing c'', that tone will be heard, and c''' may, by attention, be distinguished as sounding with it. But if the fork be placed on the first string at the proper point for the finger in playing c''', that tone will be distinctly heard, while c'' will be weak [3].

[2] This harmonic octave in the sound of the tuning-fork is a phænomenon of the second order, of the kind mentioned in Art. **112**.

[3] The sound actually heard is caused by waves in the air originating not directly from the vibrations of the string, but indirectly from those communicated from the string, through the bridge, to the sound-board. An investigation of the effect of placing a tuning-fork on the string, in which this circumstance is taken into account, is given by Helmholz (*Beilage* III). The practical results agree with those of the simpler process in the text.

These are particular cases of the general phænomenon of *resonance*, of which we shall have to speak hereafter.

131. The expression (6), or its equivalent (7), becomes inapplicable, as was shewn in Art. **125**, when the obligatory vibration is imposed at such a point that its period is any aliquot part of the natural period of vibration of the portion of string considered.

This inconvenience may be avoided by introducing the hypothesis of resistance, under the form already employed in Art. **125**. And as the result will be useful in a subsequent problem, we shall give the process as briefly as possible. We shall neglect altogether the *natural* vibrations, which are soon extinguished, and with which we shall have no concern in the application to be made hereafter.

Let l be the length of the whole string. Then, putting c instead of $2k$ in the differential equation (2′) of Art. **125**, we have

$$\frac{d^2y}{dt^2} + c\,\frac{dy}{dt} = a^2\,\frac{d^2y}{dx^2};$$

and we are to obtain a solution of this equation satisfying the conditions

$$y = p\sin nt + q\cos nt \text{ when } x = b, \text{ and } y = 0 \text{ when } x = 0.$$

For this purpose we may assume

$$y = u\sin nt + v\cos nt,$$

u and v being functions of x to be determined.

Substituting this value of y in the differential equation, and equating to 0 the coefficients of $\sin nt$ and $\cos nt$, we find the two conditions

$$\left.\begin{aligned}(a^2D^2+n^2)\,u + cnv = 0,\\ (a^2D^2+n^2)\,v - cnu = 0,\end{aligned}\right\} \tag{I}$$

in which D stands for $\dfrac{d}{dx}$.

Eliminating v we obtain

$$\{(a^2D^2+n^2)^2 + c^2n^2\}\,u = 0;$$

and therefore

$$u = A\epsilon^{\alpha_1 x} + B\epsilon^{\alpha_2 x} + C\epsilon^{\alpha_3 x} + D\epsilon^{\alpha_4 x},$$

where α_1, α_2, α_3, α_4 are the four roots of the equation

$$(a^2\alpha^2+n^2)^2 + c^2n^2 = 0;$$

that is, the four values of the expression

$$\pm\frac{n}{a}\sqrt{-1}\left\{1 \pm \frac{c}{n}\sqrt{-1}\right\}^{\frac{1}{2}}. \tag{a}$$

To put this result in a convenient form, assume

$$\frac{c}{n} = \tan\psi, \qquad \tan\frac{\psi}{2} = \beta, \text{ then}$$

$$1 \pm \frac{c}{n}\sqrt{-1} = \frac{1}{\cos\psi}(\cos\psi \pm \sqrt{-1}\sin\psi);$$

whence, employing De Moivre's theorem, and observing that

$$\frac{\cos\frac{1}{2}\psi}{\sqrt{\cos\psi}} = \frac{1}{\sqrt{1-\beta^2}},$$

we find for the values of the expression (a) those of

$$\frac{n}{a\sqrt{1-\beta^2}}(\pm\sqrt{-1} \pm \beta)$$

and the above value of u becomes (after putting A, B instead of $A\sqrt{-1}$, $B\sqrt{-1}$)

$$u = \sin\theta\,(A\epsilon^{\beta\theta} + B\epsilon^{-\beta\theta}) + \cos\theta\,(C\epsilon^{\beta\theta} + D\epsilon^{-\beta\theta}),$$

in which, $\qquad \theta = \dfrac{nx}{a\sqrt{1-\beta^2}}.$

The differential equation for v is of the same form as that for u. Hence the value of v will only differ from that of u in having different constants A', B', C', D' instead of A, B, C, D. But these eight constants cannot be all independent of one another, since the solution of the simultaneous equations (I) cannot contain more than four arbitrary constants. In fact, on substituting the values of u and v in those equations, we obtain the relations $\qquad A' = C, \quad B' = -D, \quad C' = -A, \quad D' = B.$

Moreover, since $y = 0$ when $x = 0$, for all values of t, we must have $u = 0$ and $v = 0$ when $\theta = 0$; hence

$$C + D = 0, \quad \text{and} \quad C' + D' = 0,$$

from which it follows that $A = B$. Finally, therefore, changing A and C into $\frac{1}{2}A$ and $\frac{1}{2}C$, we may write the values of u and v thus:

$$y = (A\sigma\sin\theta + C\delta\cos\theta)\sin nt$$
$$+ (C\sigma\sin\theta - A\delta\cos\theta)\cos nt; \quad \text{where}$$

$$\sigma = \frac{\epsilon^{\beta\theta} + \epsilon^{-\beta\theta}}{2}, \qquad \delta = \frac{\epsilon^{\beta\theta} - \epsilon^{-\beta\theta}}{2}.$$

It remains to determine A and C so that the above expression may become identical with

$$p\sin nt + q\cos nt$$

when $x = b$, that is, when $\theta = \dfrac{nb}{a\sqrt{1-\beta^2}}.$

Putting therefore ϕ for this particular value of θ, and σ_0, δ_0 for the corresponding values of σ, δ, we have the two equations

$$A\,\sigma_0 \sin\phi + C\,\delta_0 \cos\phi = p,$$
$$C\,\sigma_0 \sin\phi - A\,\delta_0 \cos\phi = q,$$

whence

$$A\,(\sigma_0^2 \sin^2\phi + \delta_0^2 \cos^2\phi) = p\,\sigma_0 \sin\phi - q\,\delta_0 \cos\phi,$$
$$C\,(\sigma_0^2 \sin^2\phi + \delta_0^2 \cos^2\phi) = q\,\sigma_0 \sin\phi + p\,\delta_0 \cos\phi;$$

and the values of A and C thus determined are to be introduced in the above expression for y.

The result may be conveniently expressed thus:

$$\text{put} \quad \frac{q\sigma_0 \sin\phi + p\delta_0 \cos\phi}{p\sigma_0 \sin\phi - q\delta_0 \cos\phi} = \tan\Phi;$$

$$\text{then} \quad y = \frac{(p^2+q^2)^{\frac{1}{2}}}{(\sigma_0^2 \sin^2\phi + \delta_0^2 \cos^2\phi)^{\frac{1}{2}}} \{\sigma \sin\theta \sin(nt+\Phi) + \delta \cos\theta \cos(nt+\Phi)\}. \quad \text{(II)}$$

This gives the exact solution of the problem; that is, it determines the motion of the string from $x = 0$ to $x = b$. The motion of the remaining part of the string will be given by putting $l - x$ for x, and $l - b$ for b, in (II). The radical in the denominator must evidently be understood to have the same sign as $\sin\phi$, in order that this value of y may agree with that found, as in Art. **126**, when resistance is neglected, or $\beta = 0$. The period of the obligatory vibration is $\frac{2\pi}{n}$, and that of the natural vibration of the part of the string from $x = 0$ to $x = b$ is $\frac{2b}{a}$; hence, if the former period be any aliquot part of the latter, or $\frac{b}{a} = \frac{i\pi}{n}$, the value of $\phi\left(= \frac{nb}{a\sqrt{1-\beta^2}}\right)$ becomes $\frac{i\pi}{\sqrt{1-\beta^2}}$, so that, β being supposed very small, and δ_0 being of the same order as β, the denominator in the expression (II) becomes very small; but it cannot vanish for any value of ϕ, and therefore this expression always gives a finite value for y.

The expression (II) is however too complicated for use in the problem for the sake of which we have obtained it. (See Art. **138**.) We shall therefore neglect β in every part of that expression except where it is essential to retain it. Now $\beta = 0$ gives $\sigma = 1$, $\sigma_0 = 1$, $\delta = 0$, $\delta_0 = 0$, $\tan\Phi = \frac{q}{p}$. But we must retain the term $\delta_0^2 \cos^2 \phi$ in the denominator, in order that y may not become infinite when $\sin \phi = 0$. We thus obtain

$$y = \sin\theta \, \frac{p \sin nt + q \cos nt}{(\sin^2 \phi + \delta_0^2 \cos^2 \phi)^{\frac{1}{2}}} \qquad \text{(III)}$$

as an approximate expression, which sensibly coincides with that obtained on the supposition of no resistance, for all but very small values of ϕ, but agrees with experiment in giving a large but not infinite value for y when $\sin \phi = 0$, or $b = \dfrac{i \pi a}{n}$. (It must be remembered that, in (III), $\theta = \dfrac{nx}{a}$ and $\phi = \dfrac{nb}{a}$).

If the vibration imposed at the point where $x = b$ were not of the simple harmonic kind, then instead of the right-hand member of (III) we should have a series of analogous terms, as in Art. **129**.

132. Problem 2. *To find the motion of the string when a given point of it is subject to a given finite periodic pressure, in a direction at right angles to* AB, *resistance being neglected.*

Suppose that the pressure is applied at a distance b from the end A, and that at the time t it is $p \sin nt$, p being a given finite constant force.

To solve this problem we must recur to the fundamental equation (see (1), Art. **121**)

$$dm \frac{d^2 y}{dt^2} = T d\left(\frac{dy}{dx}\right),$$

of which the meaning is that the force on any element dm is the difference of the transverse components of the tension at its two ends.

Now in general the change of direction of the string is continuous, so that $d\left(\dfrac{dy}{dx}\right)$ is infinitesimal. But if the element dm contain the point at which the finite pressure is applied, there will, or at any rate *may*, be a *sudden* change of direction at that point, so that the values of $\dfrac{dy}{dx}$ at two points on different sides of it, however near, may differ by a finite quantity. Hence, *for that particular element* we must write $\Delta\left(\dfrac{dy}{dx}\right)$ instead of $d\left(\dfrac{dy}{dx}\right)$; and since it is acted on by the given pressure in addition to the difference of component tensions, we must have, *for that element,*

$$dm \cdot \frac{d^2y}{dt^2} = T \,\Delta \left(\frac{dy}{dx}\right) + p \sin nt,$$

and this equation must subsist, *however small we take dm;* hence in the limit it becomes

$$T.\Delta\left(\frac{dy}{dx}\right) + p \sin nt = 0, \tag{8}$$

which is a condition that must be satisfied for the point at which $x = b$.

At all other points the usual differential equation (3) must be satisfied.

Now we may satisfy (3) with the conditions that $y = 0$ for $x = 0$ and $x = l$, and at the same time allow the possibility of a finite change in the value of $\frac{dy}{dx}$ when x passes through the value b, by assuming

$$y = \sin mx\,(A \cos mat + B \sin mat) \text{ from } x = 0 \text{ to } x = b,$$
$$y = \sin m\,(l-x)\,(A' \cos mat + B' \sin mat) \text{ from } x = b \text{ to } x = l.$$

The condition that these values of y must coincide when $x = b$, gives

$$A \sin mb = A' \sin m\,(l-b),$$
$$B \sin mb = B' \sin m\,(l-b).$$

Moreover it is evident that it will be impossible to satisfy (8) without taking $ma = n$.

Hence we may put

$$y = \sin \frac{n\,(l-b)}{a} \sin \frac{nx}{a}\,(C \cos nt + D \sin nt),$$

from $x = 0$ to $x = b$, and

$$y = \sin \frac{nb}{a} \sin \frac{n\,(l-x)}{a}\,(C \cos nt + D \sin nt)$$

from $x = b$ to $x = l$;

where C and D are constants to be determined by the condition (8).

Now if we put $x = b$ in the values of $\frac{dy}{dx}$ derived from the two expressions above given, and then subtract the first result from the second, we obtain

$$\Delta\left(\frac{dy}{dx}\right) = -\frac{n}{a} \sin \frac{nl}{a}\,(C \cos nt + D \sin nt);$$

so that, in order to satisfy (8), we must have $C = 0$, and

$$D = \frac{a}{n} \cdot \frac{p}{T \sin \frac{nl}{a}}; \quad \text{whence}$$

$$\left.\begin{aligned} y &= \frac{a}{n}\frac{p}{T} \cdot \frac{\sin \frac{n(l-b)}{a} \sin \frac{nx}{a}}{\sin \frac{nl}{a}} \sin nt \quad (x = 0 \text{ to } x = b), \\ y &= \frac{a}{n}\frac{p}{T} \cdot \frac{\sin \frac{nb}{a} \sin \frac{n(l-x)}{a}}{\sin \frac{nl}{a}} \sin nt \quad (x = b \text{ to } x = l). \end{aligned}\right\} \quad (9)$$

These expressions determine the *forced* vibration of the string; and it is evident that we may add the terms representing the natural vibration, viz.

$$\sum_{i=1}^{i=\infty} \sin \frac{i\pi x}{l} \left\{ A_i \cos \frac{i\pi a t}{l} + B_i \sin \frac{i\pi a t}{l} \right\}. \qquad (9')$$

(The above expressions for the forced vibration lead to a result which at first sight appears paradoxical. Suppose, namely, that $\sin \frac{nb}{a} = 0$, which will happen if the period of the forced vibration coincide with that of the natural vibration of the first portion of the string, considered as fixed at both ends. Then the value of y in the other portion will be always 0, or that portion will remain at rest, if the constants A_i, B_i be all 0.

The explanation is merely that in this case the periodic pressure is equal to that which the string would exert on a fixed point at the same place, if the portion on one side of it were vibrating naturally, and the other portion at rest.)

133. We have introduced this problem for the sake of the use which may be made of it in finding approximately the character of the vibrations excited in a pianoforte string by the blow of the hammer. For this purpose we shall adopt the hypothesis proposed by Helmholz, (the reason for which is explained below, Art. **134**,) namely, that the pressure of the hammer on the string during contact may be represented by an expression of the form $p \sin nt$, but that it lasts only during half a period, viz. from $t = 0$ to $t = \frac{\pi}{n}$. The breadth of the hammer is neglected.

Now if, in the solution of the above problem, we suppose the constants A_i, B_i to be so determined that at some one instant when the pressure vanishes, say when $t = 0$, the values of y and of $\frac{dy}{dt}$ shall vanish at all points of the string, then the subsequent motion will be the same as if the pressure had only begun to exist, and to disturb the string, at that instant. If then we find the values of y and $\frac{dy}{dt}$ at the end of the first half-period of pressure, that is, when $t = \frac{\pi}{n}$, these will give the initial data for calculating the natural vibrations which would follow if the pressure then ceased to exist, as we suppose it to do in the case of the pianoforte.

Now referring to the general expression (9) + (9′) obtained for y, we see that the condition $y = 0$ when $t = 0$ gives $A_i = 0$ for all values of i; and the condition $\frac{dy}{dt} = 0$ gives

$$\pi \frac{a}{l} \Sigma\, i B_i \sin \frac{i\pi x}{l}$$

$$= -\frac{ap}{T} \frac{\sin \frac{n(l-b)}{a} \sin \frac{nx}{a}}{\sin \frac{nl}{a}} \quad (x = 0 \text{ to } x = b)$$

$$= -\frac{ap}{T} \frac{\sin \frac{nb}{a} \sin \frac{n(l-x)}{a}}{\sin \frac{nl}{a}} \quad (x = b \text{ to } x = l);$$

whence, multiplying each side by $\sin \frac{i\pi x}{l} dx$, and integrating from $x = 0$ to $x = l$, we have

$$-\frac{i\pi T}{2p} \sin \frac{nl}{a} B_i = \sin \frac{n(l-b)}{a} \int_0^b \sin \frac{nx}{a} \sin \frac{i\pi x}{l} dx$$

$$+ \sin \frac{nb}{a} \int_b^l \sin \frac{n(l-x)}{a} \sin \frac{i\pi x}{l} dx. \qquad (10)$$

The value of the right-hand member of this equation will be found to be

$$-\frac{nal^2}{n^2 l^2 - a^2 i^2 \pi^2} \sin \frac{nl}{a} \sin \frac{i\pi b}{l},$$

and therefore

$$B_i = \frac{2p}{i\pi T} \cdot \frac{n a l^2}{n^2 l^2 - a^2 i^2 \pi^2} \sin \frac{i\pi b}{l}. \tag{11}$$

Now if, in the general expression (9) + (9′) for y, we put $A_i = 0$, we find, for the instant when $nt = \pi$,

$$y = \Sigma B_i \sin \frac{i\pi x}{l} \sin \frac{i\pi^2 a}{nl},$$

$$\frac{dy}{dt} = \frac{a\pi}{l} \Sigma i B_i \sin \frac{i\pi x}{l} \cos \frac{i\pi^2 a}{nl} + y';$$

where the value of y' is

$$-\frac{ap}{T} \frac{\sin \frac{n(l-b)}{a} \sin \frac{nx}{a}}{\sin \frac{nl}{a}} \text{ from } x = 0 \text{ to } x = b,$$

$$\text{and } -\frac{ap}{T} \frac{\sin \frac{nb}{a} \sin \frac{n(l-x)}{a}}{\sin \frac{nl}{a}} \text{ from } x = b \text{ to } x = l;$$

and these are the initial values of y and $\frac{dy}{dt}$ with which we are to calculate the subsequent natural vibrations.

If then we begin to reckon t afresh from this instant, and assume

$$y = \Sigma \sin \frac{i\pi x}{l} \left(C_i \cos \frac{i\pi a t}{l} + D_i \sin \frac{i\pi a t}{l} \right); \tag{12}$$

and compare the above initial values with those derived from this expression, we find

$$C_i = B_i \sin \frac{i\pi^2 a}{nl},$$

$$\frac{a\pi}{l} \Sigma i D_i \sin \frac{i\pi x}{l} = \frac{a\pi}{l} \Sigma i B_i \cos \frac{i\pi^2 a}{nl} \sin \frac{i\pi x}{l} + y';$$

and multiplying the last equation by $\sin \frac{i\pi x}{l} dx$, and integrating from $x = 0$ to $x = l$, (observing that (10) gives

$$\int_0^l y' \sin \frac{i\pi x}{l} dx = \frac{a i\pi}{2} B_i),$$

we obtain

$$D_i = B_i \left(1 + \cos \frac{i\pi^2 a}{nl} \right)$$

and therefore the value (12) of y becomes, after a slight reduction,

$$y = 2\sum_{i=1}^{i=\infty} B_i \cos\frac{i\pi^2 a}{2nl} \sin\frac{i\pi x}{l} \sin\frac{i\pi a}{l}\left(t+\frac{\pi}{2n}\right); \quad (13)$$

which equation determines the motion of the string after the pressure has ceased to act.

The amplitude[4] of the component vibration which gives the ith harmonic tone is therefore $2B_i \cos\frac{i\pi^2 a}{2nl}$; or, B_i being replaced by its actual value (11), the amplitude is

$$\frac{4p}{\pi T}\cdot\frac{nal^2}{i(n^2l^2 - i^2a^2\pi^2)} \sin\frac{i\pi b}{l} \cos\frac{i\pi^2 a}{2nl}. \quad (14)$$

(This expression will be seen to agree with the result obtained in a different manner by Helmholz, (*Beilage* IV. equation (12*a*),) if it be observed that p, T, i, n in (14) correspond to Helmholz's A, S, n, m; and that Helmholz's A_n is the amplitude only of the negative wave, and is therefore *half* the amplitude of the complete vibration.)

134. In order to understand the results deducible from the expression (14) we must recur to the hypothesis made above concerning the law of pressure during the contact of the hammer with the string. That hypothesis is founded on the assumption that the impact may be assimilated to that of an elastic body upon a hard fixed obstacle. On this supposition let λ be the length of the hammer, μ its mass, μk^2 its moment of inertia about the axis on which it turns, θ the angle through which it has turned from rest at the time t, θ_0 and θ'_0 the values of θ and $\frac{d\theta}{dt}$ at the instant when contact begins, and at which we will suppose $t = 0$; then the pressure will be $\lambda q(\theta - \theta_0)$ during contact, q being a constant depending on the elasticity

[4] The meaning of 'amplitude' has been before defined (Art. 48) with reference to the harmonic vibration of a *point*. In the case of the harmonic vibration of a string, expressed by the equation

$$y = L\sin\frac{i\pi x}{l}\sin\left(\frac{i\pi a t}{l} + M\right),$$

or by the equivalent form

$$y = \sin\frac{i\pi x}{l}\left(C\cos\frac{i\pi a t}{l} + D\sin\frac{i\pi a t}{l}\right);$$

the amplitude may be defined as the maximum displacement from the position of equilibrium. This maximum displacement is evidently equal to L, or to $(C^2 + D^2)^{\frac{1}{2}}$, and occurs at points which bisect the nodal intervals.

of the material of which the head of the hammer is made; and we shall have, during contact,

$$\frac{d^2\theta}{dt^2} = -\frac{\lambda^2 q(\theta - \theta_0)}{\mu k^2};$$

and if this be integrated in the usual way, and the constants determined by the conditions $\theta = \theta_0$, $\frac{d\theta}{dt} = \theta'_0$ when $t = 0$, the result is

$$\theta - \theta_0 = \frac{\theta_0'}{n} \sin nt$$

where $$n = \frac{\lambda}{k}\sqrt{\frac{q}{\mu}};$$

and the pressure during contact is therefore

$$\theta'_0 k\sqrt{\mu q} \,.\, \sin nt,$$

and the duration of contact is

$$\frac{\pi}{n} = \pi\frac{k}{\lambda}\sqrt{\frac{\mu}{q}}.$$

Hence the value of p in (14) is $\theta'_0 k\sqrt{\mu q}$, which depends on the velocity of the hammer at the beginning of impact, as well as on its weight, material and form. But the value of n, which determines the duration of contact, depends only on the latter circumstances, and not on the velocity.

135. Referring now to the expression (14) we see that it vanishes if $\sin\frac{i\pi b}{l} = 0$; that is, if $ib = ml$, m being any integer. This shews that if the blow of the hammer be applied at any one of the points which divide the string into i equal parts, all the harmonic component tones which would have nodes at those points are extinguished. (We found a similar result on the supposition that the sound was excited by plucking the string at a given point. See Art. **119**.)

In general the quality of the note produced, which is determined by the comparative strength of its different component tones, is independent of the momentum of the blow, (which only affects the value of the coefficient p,) but depends upon the two ratios $\frac{b}{l}$ and $\frac{a}{nl}$, that is, upon the place at which the blow is struck, and upon the ratio of the duration of contact to the period of the fundamental tone of the string $\left(\text{viz. } \frac{\pi a}{2nl}\right)$. If we put ν for this latter ratio, the expression (14) may be written in the form

$$\frac{8pl}{T} \cdot \frac{\nu}{i(1-4i^2\nu^2)} \sin\frac{i\pi b}{l} \cos(i\pi\nu).$$

The value of ν depends, *caeteris paribus*, on the coefficient of elasticity q, and becomes very small if q be very great, that is, if the hammer is of a hard unyielding material. But the product $p\nu$ is seen, on reference to the values of p and ν, to be independent of q, so that the above expression for the amplitude becomes

$$\frac{A}{i(1-4i^2\nu^2)} \sin\frac{i\pi b}{l} \cos(i\pi\nu);$$

where A depends upon the weight, form, and velocity of the hammer, but not upon its elasticity. If we suppose the hammer absolutely hard, or $\nu = 0$, then the expression becomes $\frac{A}{i}\sin\frac{i\pi b}{l}$.

A table of results calculated on different suppositions as to the place of the blow and the value of ν may be seen in Helmholz, p. 135.

It appears that in general the effect of diminishing ν is to increase the strength of some of the higher harmonies as compared with that of the fundamental tone.

In numerical applications it must be remembered that the intensity of the tone is supposed to be proportional (for each harmonic component) to the square of the amplitude multiplied by i^2. See Problem 3.

136. Problem 3. *To find the energy of a string vibrating naturally.*

First, suppose the vibrations are in one plane, and such that the note produced is simply the i^{th} harmonic component. Then the form of the string at any time may be represented by the equation

$$y = A\sin\frac{i\pi x}{l}\sin\left(\frac{i\pi a t}{l} + \alpha\right). \tag{15}$$

The total energy at any time consists, as we know from general principles, of two parts, kinetic and potential, of which the sum is constant. The kinetic energy is that due to the motion of the string, and is measured by half the *vis-viva*. The potential energy is that due to the deviation of the string from the form of equilibrium, and is entirely converted into kinetic energy whenever the string is passing through the position of equilibrium, that is, whenever

$$\sin\left(\frac{i\pi a t}{l} + \alpha\right) = 0.$$

Hence the total energy at any time is equal to the kinetic energy at any one of those particular instants.

Now (15) gives, when $\sin\left(\frac{i\pi a t}{l} + a\right) = 0$,

$$\frac{dy}{dt} = \pm i \frac{\pi a}{l} A \sin \frac{i\pi x}{l};$$

so that, if ρ be the mass of a unit of length, the kinetic energy at that instant is

$$\tfrac{1}{2} \rho i^2 \left(\frac{\pi a}{l}\right)^2 A^2 \int_0^l \left(\sin \frac{i\pi x}{l}\right)^2 dx = \tfrac{1}{4} \rho l i^2 \left(\frac{\pi a}{l}\right)^2 A^2;$$

which is the required value of the total energy at any time. We may transform this expression as follows: let M be the mass of the whole string, and τ_i the period of the vibration. Then

$$\rho l = M, \quad \text{and} \quad a\tau_i = \frac{2l}{i}, \quad \text{or} \quad \frac{ia}{l} = \frac{2}{\tau_i};$$

hence the above expression becomes

$$\pi^2 M \frac{A^2}{\tau_i^2}; \tag{16}$$

from which we learn that in this case *the energy is proportional to the product of the mass of the whole string, the square of the amplitude, and the square of the number of vibrations in a unit of time.*

Next, suppose the vibrations still in one plane, but of the most general kind; then, at any time,

$$y = \Sigma \sin \frac{i\pi x}{l} \left(A_i \cos \frac{i\pi a t}{l} + B_i \sin \frac{i\pi a t}{l}\right).$$

In this case the string, in general, never passes through the form of equilibrium, and the potential energy is therefore never entirely converted into kinetic.

Let us consider the string at the instant when $t = 0$: at that instant

$$y = \Sigma A_i \sin \frac{i\pi x}{l}, \tag{17}$$

and

$$\frac{dy}{dt} = \frac{a\pi}{l} \Sigma i B_i \sin \frac{i\pi x}{l}; \tag{18}$$

and the total energy is

$$K + P,$$

where K is the kinetic energy due to the motion expressed by (18), and P is the potential energy due to the form expressed by (17).

The value of K is easily found, for

$$K = \tfrac{1}{2}\rho\left(\frac{a\pi}{l}\right)^2 \int_0^l \left(\Sigma\, i\, B_i \sin\frac{i\pi x}{l}\right)^2 dx;$$

and if we suppose the square of the series within brackets to be developed, we see at once that all the terms will be destroyed by the integration except those comprised in the series

$$\Sigma\left(i^2 B_i^2 \left(\sin\frac{i\pi x}{l}\right)^2\right);$$

and these give, on integration, $\frac{1}{2} l\, \Sigma\, i^2 B_i^2$. Hence we have

$$K = \tfrac{1}{4}\rho\, l \left(\frac{a\pi}{l}\right)^2 \Sigma\, i^2 B_i^2. \tag{19}$$

To find the value of P, we may proceed as follows:

Suppose the string to be put in the form (17), and then left to itself without any initial velocity. After the lapse of any time t', its total energy will still be equal to P, but will have been partly converted into kinetic, so that

$$P = K_1 + P_1,$$

where K_1 is the kinetic energy at the instant in question, and P_1 is the potential energy due to the form at the same instant. Suppose then the string to be brought to rest in that form, and again left to itself for a time t'; we should have in like manner

$$P_1 = K_2 + P_2,$$

and so on successively; thus

$$P = K_1 + K_2 + K_3 + \dots \textit{ ad infin.},$$

where K_1, K_2, ... are the kinetic energies which the string would have after successive intervals of time equal to t', if, beginning with the form (17), it were left to itself for a time t', then brought to rest in its actual form, and left to itself again, and so on successively.

Now the string, left to itself at a given instant in the form (17), will vibrate (see Art. **97**) so that at any time (reckoned from that instant)

$$y = \Sigma\, A_i \sin\frac{i\pi x}{l} \cos\frac{i\pi a t}{l};$$

and at the end of any time t', if we put $\dfrac{\pi a t'}{l} = \theta$, we shall have

$$y = \Sigma\, A_i \cos i\theta \,.\, \sin\frac{i\pi x}{l}; \tag{20}$$

$$\frac{dy}{dt} = -\frac{\pi a}{l} \Sigma i A_i \sin i\theta . \sin \frac{i\pi x}{l}; \tag{21}$$

and K_1 is the kinetic energy obtained from (21) in the same way as K was from (18); hence

$$K_1 = \tfrac{1}{4} \rho\, l \left(\frac{\pi a}{l}\right)^2 \Sigma\, i^2 A_i^2 (\sin i\theta)^2,$$

To find K_2 we have to proceed in the same way, merely assuming (20) instead of (17) as the initial form; thus

$$K_2 = \tfrac{1}{4} \rho\, l \left(\frac{\pi a}{l}\right)^2 \Sigma\, i^2 A_i^2 (\cos i\theta)^2 (\sin i\theta)^2;$$

and so on successively, the value of K_{n+1} being always deduced from that of K_n by changing A_i into $A_i \cos i\theta$. Therefore

$$P = K_1 + K_2 + \ldots$$
$$= \tfrac{1}{4} \rho l \left(\frac{\pi a}{l}\right)^2 \Sigma\, i^2 A_i^2 (\sin i\theta)^2 (1 + (\cos i\theta)^2 + (\cos i\theta)^4 \ldots \textit{ad inf.})$$

and, the series within brackets being equivalent to

$$\frac{1}{1-(\cos i\theta)^2} = \frac{1}{(\sin i\theta)^2},$$

we have finally[5]

$$P = \tfrac{1}{4} \rho\, l \left(\frac{\pi a}{l}\right)^2 \Sigma\, i^2 A_i^2.$$

Adding the value of P thus found to that of K (19), we obtain the required expression for the total energy (E) of the string, viz.

$$E = \tfrac{1}{4} \rho\, l \left(\frac{\pi a}{l}\right)^2 \Sigma\, i^2 (A_i^2 + B_i^2)$$

$$= \pi^2 M \Sigma \frac{A_i^2 + B_i^2}{\tau_i^2};$$

[5] The representation of P by an infinite series corresponds to the physical fact that it would require an infinite number of operations of the kind described in the text to bring the string into the condition of equilibrium. It may be observed that, if the arbitrary θ be taken incommensurable with π, the series within brackets cannot become divergent, though for infinitely large values of i it may approach infinitely near to divergence; but this will be compensated by the factor $(\sin i\theta)^2$ becoming infinitely small. If we took t' equal to half a period (or $\theta = \pi$), it is evident that the operations described would *never* bring the string to rest. In this case the factor $(\sin i\theta)^2$ would vanish, and the series within brackets become ∞. But we arrive at a true result by interpreting the product as representing 1, for this as for all other values of θ.

where, as before, M is the mass of the string, and τ_i the period of the i^{th} harmonic component vibration.

Now, observing that $A_i^2+B_i^2$ is the square of the amplitude of the component vibration, and comparing this result with (16), we see that the total energy E is the sum of the energies due to the several harmonic components.

Lastly, we will take the most general case, in which the vibration is not in one plane. The displacement of any point in the string at a distance x from one end is then compounded of two displacements y and z in planes at right angles to one another, and the whole vibration is compounded of two represented by equations

$$y = \Sigma \sin \frac{i\pi x}{l}\left(A_i \cos \frac{i\pi a t}{l} + B_i \sin \frac{i\pi a t}{l}\right),$$

$$z = \Sigma \sin \frac{i\pi x}{l}\left(A'_i \cos \frac{i\pi a t}{l} + B'_i \sin \frac{i\pi a t}{l}\right);$$

the square of the velocity at any point of which the abscissa is x, is now, when $t = 0$,

$$\left(\frac{\pi a}{l}\right)^2 \left\{\left(\Sigma\, i B_i \sin \frac{i\pi x}{l}\right)^2 + \left(\Sigma\, i B'_i \sin \frac{i\pi x}{l}\right)^2\right\};$$

and the process is the same as before, with obvious modifications which may be left to the reader.

The result is

$$E = \pi^2 M \Sigma \frac{A_i^2 + B_i^2 + A'^2_i + B'^2_i}{\tau_i^2};$$

and, as before, the total energy is the sum of the energies due to the several harmonic component vibrations in both planes.

It will be observed that the numerator in the above expression is for each harmonic vibration the sum of the squares of the amplitudes of its components in the two planes; and this sum may, by an extension of meaning, be called the square of the amplitude of the actual vibration, which, for a given point, is in general elliptic.

137. The value of E found in the last article, is the equivalent of the work which would have to be done, if the string were at rest, in order to put it into its actual form and state of motion. And it appears natural to take this as the measure of the strength or intensity of the note produced. But the propriety of this definition cannot be absolutely demonstrated by experiment, because, although the ear can judge with great accuracy which of two notes is the louder, when both have the same quality and the same pitch, it cannot form a precise judgment as to the *ratio* of intensity. On the other hand, when two notes have the same quality but differ moderately in pitch, the ear can still decide with some certainty whether they are or are not of *equal* intensity, and if not, which is the louder; and it might perhaps be possible to arrange an experiment in which a series of notes should have the same quality and equal intensities according to the theoretical measure, and the ear would judge whether equality of loudness subsisted at the same time. (The definition of *quality* will be discussed in another chapter.)

138. Problem 4. *To examine the motion of a violin string under the action of the bow.*

This problem is much more difficult than that of the pianoforte string, because the force exercised by the bow upon the string is determined by circumstances which seem to defy calculation, and we can hardly make any plausible hypothesis *à priori.* We are obliged therefore to have recourse to observation, and endeavour to determine experimentally some characteristics of the motion from which the analytical representation of it may be deduced.

In the first place then it may be easily verified by any one with a practised ear, that when the bow is drawn across the string at any point of aliquot division, no component tone which would (if existing alone) have a node at that point, is heard in the note produced. (In order however to extinguish these tones, it is necessary that the coincidence of the point of application of the bow with the node should be *exact.* A very small deviation reproduces the missing tones with considerable strength.) The other facts to which we shall have to refer are ascertained not by the ear but by the eye. The character of the vibration of any point of the string may be observed by means of the 'vibration-microscope', the principle of which was explained in Art. **65**, and in this way Helmholz has arrived at results of which the following are the most important:

(*a.*) When the bow is applied at a point of which the distance from the bridge is an aliquot part of the string, and the point observed is one of the other nodes of the same division, the curve obtained by the imaginary unrolling of the cylinder (Art. **65**) reduces itself to a zigzag line, so that a complete vibration is represented thus—

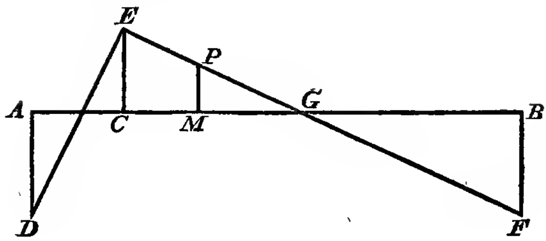

Fig. 1.

$A\,B$ represents the period (τ) of the vibration, and the ordinate $P\,M$ of any point P in the line $D\,E\,F$ represents the displacement of the observed point at the time t ($=A\,M$) reckoned from the instant of greatest negative displacement. It appears to be implied that $AD = CE$, or that the excursions on opposite sides of the position of equilibrium are equal.

It follows evidently that the velocity of the observed point is constant throughout each of the two parts (the 'swing' and the 'swang') of the vibration, but is not in general the same in each. When, however, the observed point is at the middle of the string, it is found that $A\,C = CB$, and the velocities are therefore equal.

(*b.*) If at any point we call the 'swing' that part of the vibration which is performed while the point is moving in accordance with the bow, then the velocity of the swing is less than that of the swang, if the observed point is in the same half of the string as the point of application of the bow, and greater in the contrary case. Thus, in Fig. 1, $D\,E$ would represent the swing and $E\,F$ the swang at a node in the contrary half to that in which the bow is applied. It appears probable that at the point of application the string is dragged by the bow with its own velocity during the swing.

(*c.*) When the observed point is not one of the nodes, the vibration is still represented approximately but not exactly by Fig. 1. In this case the lines DE, EF, instead of being perfectly straight, consist of a series of ripples or wavelets, though maintaining their average directions.

When the bow is applied at a point which is not a node, the character of the vibrations has not been satisfactorily made out. (Helmholz, p. 139, &c.) (The reader will observe that we are here using the word *node* to signify not an *actual* node, but a point which would be a node if the corresponding component vibration existed alone.)

These results have been confirmed by Professor Clifton, who observed the vibration-curves of points on the string by means of revolving mirrors. (On the principle of this method of observation, see Note at the end of this chapter).

139. Assuming the facts above stated, let us suppose that the length of the string is l, and that the bow is applied at a point which we will call Q, at a distance b from the bridge. Now, whatever be the character of the actual vibration of Q, we know that it can be expressed by means of Fourier's theorem in the form

$$\Sigma\left(C_i \cos\frac{2i\pi t}{\tau} + D_i \sin\frac{2i\pi t}{\tau}\right); \qquad (22)$$

where τ is the period of the vibration, which must be that of the natural vibration of the string, since we assume that the fundamental note is produced; hence $\tau = \dfrac{l}{2a}$.

Moreover it is evident that if the actual vibration of Q were known, we might suppose it to become *obligatory*, and the motion of the rest of the string would remain unaltered.

We shall therefore in the first place assume that the series of coefficients C_i, D_i are known, and that (22) is the obligatory value of y at Q.

Then (see equation (III), Art. **131**, and the remark at the end of that article) the vibration at any point of the string from $x = 0$ to $x = b$ will be approximately represented by the equation

$$y = \sum_{i=1}^{i=\infty} \frac{\sin\theta_i}{(\sin^2\phi_i + \epsilon_i^2\cos^2\phi_i)^{\frac{1}{2}}} \times \left(C_i \cos 2i\pi\frac{t}{\tau} + D_i \sin 2i\pi\frac{t}{\tau}\right); \qquad (23)$$

where, in the present case,

$$\theta_i = \frac{2i\pi x}{\tau a} = \frac{i\pi x}{l}, \qquad \phi_i = \frac{i\pi b}{l},$$

and ϵ_i^2 is a small quantity depending on the resistance. And since this formula is not altered by changing x into $l-x$ and b into $l-b$, it will hold good for the whole length of the string.

140. The facts above stated ((*a*) (*b*) (*c*)) have been ascertained only in the case in which the point Q, at which the bow is applied, is a node.

We must therefore assume this; and in order to determine C_i, D_i, we shall further assume that the vibration of Q, represented by (22), is of the same kind as that observed at other nodes; so that (22) must give the value of the ordinate at any point P in a line such as DEF (Fig. 1), if the abscissa AM be taken proportional to t.

We shall have then $AB=\tau$; let $AC=\tau_0$, and $CE=\beta$, so that β is the amplitude of the vibration at Q, τ_0 is the duration of the 'swing,' and $\tau-\tau_0$ of the 'swang,' at the same point.

Now the problem of representing a locus such as DEF by means of a periodic series with period τ, has been already solved in Art. **76.** In order to make use of equation (3) of that Article, it is evidently only necessary to omit the constant term, and change $b, \quad \lambda, \quad a, \quad x$

into $2\beta, \quad \tau, \quad \tau_0, \quad t.$

We thus obtain for the ordinate the value

$$\frac{2\beta\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\Sigma\frac{1}{i^2}\sin\frac{i\pi\tau_0}{\tau}\sin\frac{2i\pi}{\tau}\left(t-\frac{\tau_0}{2}\right); \qquad (24)$$

and the values of C_i, D_i are to be so taken that the expression (22) shall be identical with this. It is unnecessary to write down these values, as it is easily seen that when they are introduced in (23) that equation will become

$$y=\frac{2\beta\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\frac{\sin i\pi\frac{\tau_0}{\tau}\sin\theta_i}{(\sin^2\phi_i+\epsilon_i^2\cos^2\phi_i)^{\frac{1}{2}}}\sin\frac{2i\pi}{\tau}\left(t-\frac{\tau_0}{2}\right). \qquad (25)$$

141. In this equation it is to be remembered that θ_i, ϕ_i are abbreviations for $\frac{i\pi x}{l}$, $\frac{i\pi b}{l}$. In order however that it may determine completely the value of y at every point of the string, it is necessary that the value of the ratio $\frac{\tau_0}{\tau}$ should be known. Now it has been already stated as a fact of experiment (Art. **138**) that every component tone is extinguished which would have a node at the point of application of the bow; that is, every component of which the period is an aliquot part of the period of vibration of a string of length b. Hence the value of y (25) ought to vanish when $\frac{2b}{a}$ is a multiple of $\frac{\tau}{i}$; now $b=\frac{l\phi_i}{i\pi}$, and $\tau=\frac{2l}{a}$, so that y ought to vanish when $\frac{2l\phi_i}{i\pi a}$ is a multiple of $\frac{2l}{ia}$, or when ϕ_i is a multiple of π. Therefore $\sin i\pi\frac{\tau_0}{\tau}$ ought to vanish when $\sin\phi_i$ vanishes, and the simplest hypothesis we can make is that

$$\sin\frac{i\pi\tau_0}{\tau} = \pm\sin\phi_i = \pm\sin\frac{i\pi b}{l}.$$

This may be satisfied either by

$$\frac{\tau_0}{\tau} = \frac{b}{l}, \quad \text{or} \quad \frac{\tau_0}{\tau} = \frac{l-b}{l};$$

but if t (in Fig. 1) is reckoned from an instant at which Q begins to follow the bow, so that the positive direction of y is that of the motion of the bow, the latter supposition must be adopted, because τ_0 ought to be greater than $\tau-\tau_0$. Then we shall have

$$\frac{\sin\frac{i\pi\tau_0}{\tau}}{\sin\frac{i\pi b}{l}} = -\cos i\pi,$$

and since $$-\cos i\pi \,.\, \sin\frac{i\pi x}{l} = \sin\frac{i\pi(l-x)}{l},$$

if we now agree to measure x from the other end of the string, which is equivalent to changing $l-x$ into x, (25) will be reduced to the form

$$y = \frac{2\beta\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\frac{\sin\phi_i\sin\theta_i}{(\sin^2\phi_i+\epsilon_i^2\cos^2\phi_i)^{\frac{1}{2}}}\sin\frac{2i\pi}{\tau}\left(t-\frac{\tau_0}{2}\right). \quad (26)$$

In this equation the factor $\dfrac{\sin\phi_i}{(\sin^2\phi_i+\epsilon_i^2\cos^2\phi_i)^{\frac{1}{2}}}$ is very nearly equal to 1, for all values of i except those which make $\sin\phi_i = 0$ or very small. If we substitute 1 for this factor, we obtain the approximate equation

$$y = \frac{2\beta\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\sin\frac{i\pi x}{l}\sin\frac{2i\pi}{\tau}\left(t-\frac{\tau_0}{2}\right); \quad (27)$$

and, comparing this with (24), we see that for any particular value of x, that is, for any particular point of the string, it gives a vibration-curve of the same kind. For if we take a quantity τ' such that $\dfrac{\tau'}{\tau} = \dfrac{x}{l}$, (27) may be put in the form

$$y = \frac{2\beta\tau^2}{\pi^2\tau_0(\tau-\tau_0)}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\sin\frac{i\pi\tau'}{\tau}\sin\frac{2i\pi}{\tau}\left(t-\frac{\tau'}{2}+\frac{\tau'-\tau_0}{2}\right), \quad (28)$$

in which the part under the sign of summation can be reduced to the same form as in (24) by changing the arbitrary instant from which t is reckoned. Hence it represents a zigzag like

Fig. 1, but with a different amplitude; and the phase of the vibration at any given time varies with τ', that is, with x.

Now τ is the duration of a whole vibration, and τ' of the first part of it, or 'swing;' hence the equation $\frac{\tau'}{\tau} = \frac{x}{l}$ expresses that at any point the durations of the 'swing' and of the 'swang' are proportional to the lengths of the two parts into which that point divides the string; and Helmholz has ascertained, by observing the ratio of AC to CB in Fig. 1, that this relation actually subsists, so that the hypothesis assumed above is justified.

The equation (27) agrees with the approximate formula which Helmholz has obtained in a somewhat different manner. It fails to represent two of the observed facts, namely, (1) the extinction of those component tones which have nodes at Q, and (2) the existence of ripples in the vibration-curve when the observed point is not a node.

The more accurate formula (26) represents these facts exactly, if we consider the factor $\frac{\sin \phi_i}{(\sin^2\phi_i + \epsilon_i^2 \cos^2\phi_i)^{\frac{1}{2}}}$ to be $= 1$ (as it is very nearly) except when $\sin \phi_i = 0$, in which last case it is $= 0$. For we have already seen that the vanishing of this factor causes the extinction of the component tones in question; and the vibration-curve (27) will be modified by the disappearance of the corresponding component curves; and the effect of their disappearance will evidently be to change the zigzag of straight lines represented by (27) into a zigzag of rippled lines [6].

If we put β' for the amplitude of the vibration at any particular point, then, neglecting ϵ_i, we have

$$\frac{2\,\beta\,\tau^2}{\pi^2\,\tau_0\,(\tau - \tau_0)} = \frac{2\,\beta'\,\tau^2}{\pi^2\,\tau'\,(\tau - \tau')},$$

whence
$$\beta' = \frac{\beta\,\tau'\,(\tau - \tau')}{\tau_0\,(\tau - \tau_0)} = \frac{\beta\,x\,(l - x)}{\tau_0\,(\tau - \tau_0)} \cdot \frac{\tau^2}{l^2};$$

let P be the amplitude at the middle point of the string, then $P = \frac{1}{4}\,\frac{\beta\,\tau^2}{\tau_0\,(\tau - \tau_0)}$, and therefore

$$\beta' = \frac{4\,P}{l^2}\,x\,(l - x),$$

[6] Helmholz's Fig. 25 (p. 144) represents the vibration-curve of a point so near the end of the string, that one side of the zigzag is too steep to have ripples. But Professor Clifton has found that they are seen on both sides when the observed point is nearer to the middle of the string.

which gives the ratio of the amplitude at any point to that at the middle point.

Introducing the above value of P in (26), we obtain

$$y = \frac{8P}{\pi^2} \Sigma \frac{1}{i^2} \sin \frac{i\pi x}{l} \sin \frac{2i\pi}{\tau}\left(t - \frac{\tau_0}{2}\right), \tag{29}$$

which agrees with Helmholz's equation (3c) (*Beilage* V).

This equation (or (26)), considered as an equation between x and y, determines approximately the form of the whole string at any time. When $t - \frac{\tau_0}{2}$ is 0, or any multiple of $\frac{\tau}{2}$, y vanishes for all values of x, so that the whole string is straight at these instants.

142. At all other times the two portions of the string between its extremities and the point of greatest displacement, are *straight.* This is easily shewn as follows. It was proved in Art. **98** that the equation to a locus consisting of two straight portions AC, CB (Fig. 2)

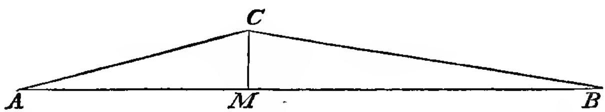

Fig. 2.

is (from $x = 0$ to $x = l$),

$$y = \frac{2y_0 l^2}{\pi^2 x_0 (l - x_0)} \Sigma \frac{1}{i^2} \sin \frac{i\pi x_0}{l} \sin \frac{i\pi x}{l}, \tag{30}$$

where $AB = l$ and x_0, y_0 are the coordinates of C (Fig. 2), reckoned (like x and y) from A. And it is evident that this equation may be made identical with (29) at any determinate time t, by taking x_0 and y_0 so that

$$\frac{2y_0 l^2}{\pi^2 x_0 (l - x_0)} = \pm \frac{8P}{\pi^2},$$

$$\sin \frac{i\pi x_0}{l} = \pm \sin \frac{2i\pi}{\tau}\left(t - \frac{\tau_0}{2}\right);$$

the same sign being taken in both.

The first of these equations shews that the locus of the vertex C of the string consists of two parabolic arcs passing through its extremities A, B, and on opposite sides of AB, belonging to the parabolas of which the equations are

$$y = \pm \frac{4P}{l^2} x (l - x).$$

The second determines the position of the vertex at the time t; and writing it in the form

$$\sin i\pi \frac{x_0}{l} = \pm \sin i\pi \frac{t - \frac{1}{2}\tau_0}{\frac{1}{2}t}$$

we see that (beginning with the instant when $t = \frac{1}{2}\tau_0$) it is satisfied by supposing x_0 to vary uniformly from 0 to l and then from l to 0 and so on successively, the time occupied by each of these successive changes being $\frac{1}{2}\tau$, and the upper and under sign being taken alternately. (This will be most clearly seen by examining the case of $i = 1$.)

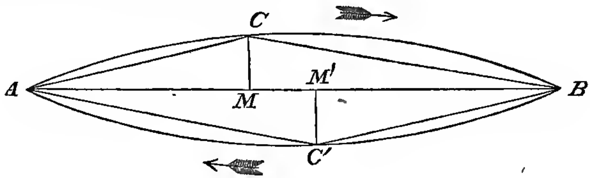

Fig. 3.

The string therefore vibrates in the following manner:

It is always divided into two straight portions, as AC, CB, or AC', $C'B$; and the vertex C describes the two parabolic arcs alternately in such a manner that the foot of the ordinate, M, moves backwards and forwards between A and B with a constant velocity. (Helmholz, *Beilage* V.)

Hitherto we have supposed the bow to be applied in the usual manner, so as to produce the fundamental note of the string. But if the point of application be taken gradually nearer to the bridge, while the bow is drawn with a somewhat quicker motion and lighter pressure, the fundamental tone becomes weaker; and ultimately a node is established at the middle of the string, the fundamental tone is extinguished, and the note produced is the octave, or first harmonic.

These changes in the character of the note are accompanied by a corresponding series of changes in the vibration-curve, which passes from the original zigzag, through a series of intermediate forms, into a similar zigzag of half the period and smaller amplitude. (See Helmholz's Fig. 26, p. 145). This phænomenon has also been observed by Professor Clifton. But it has not been submitted to mathematical analysis.

143. PROBLEM 5. *To examine the vibration of a string which is loaded with a finite mass at a given point.*

We shall assume that the weight of the mass is insignificant compared with the tension of the string, so that the vibration is modified only by its inertia; and also that its dimensions are so small that the consideration of its motion relatively to its own centre of inertia may be neglected. In other words, we shall consider it as a small but finite mass, concentrated at a point.

Let then μ be this mass, and suppose l is the length of the string, and b the distance of the point at which μ is attached from that end of the string from which x is reckoned.

As in Problem 2, we must suppose that the value of $\frac{dy}{dx}$ may undergo a sudden change when x passes through the value b; and at that point the equation

$$\mu \frac{d^2 y}{dt^2} = T \Delta \left(\frac{dy}{dx}\right) \tag{31}$$

must be satisfied. The rest of the string is subject to the usual differential equation

$$\frac{d^2 y}{dt^2} = a^2 \frac{d^2 y}{dx^2}.$$

Now we may satisfy the latter equation, together with the conditions that $y = 0$ when $x = 0$ and when $x = l$, and that the value of y must not change suddenly when $x = b$, exactly in the same way as in Problem 2. We shall therefore assume

$$y = \sin m(l-b) \sin mx \,(A \cos amt + B \sin amt), \tag{32}$$

from $x = 0$ to $x = b$, and

$$y = \sin mb \sin m(l-x)\,(A \cos amt + B \sin amt), \tag{33}$$

from $x = b$ to $x = l$.

Either of the above equations gives, for $x = b$,

$$\frac{d^2 y}{dt^2} = -a^2 m^2 \sin mb \sin m(l-b)\,(A \cos amt + B \sin amt);$$

and, taking the difference of values of $\frac{dy}{dx}$ given by the two equations when $x = b$, we find, as in Problem 2,

$$\Delta\left(\frac{dy}{dx}\right) = -m \sin ml \,.\, (A \cos amt + B \sin amt);$$

hence, in order to satisfy (31), we must have

$$\mu a^2 m \sin mb \sin m(l-b) = T \sin ml.$$

This last equation determines m, while A and B remain arbitrary unless the initial circumstances of the motion are given.

The value of a^2 is $\frac{T}{\rho}$ (Art. **123**), where ρ is the longitudinal density of the string. If then we put $\mu = \rho\lambda$, so that λ is the length of string which would have the same mass as μ, the above equation will become

$$m\lambda \sin mb \sin m(l-b) = \sin ml. \tag{34}$$

It will evidently have an infinite number of roots, and if we denote them by $m_1, m_2, \ldots$ the complete solution of the problem will be given by the equation

$$y = \sum_{i=1}^{i=\infty} X_i (A_i \cos a m_i t + B_i \sin a m_i t); \tag{35}$$

in which

$$X_i = \begin{cases} \sin m_i (l-b) \sin m_i x & (x = 0 \text{ to } x = b), \\ \sin m_i b \sin m_i (l-x) & (x = b \text{ to } x = l), \end{cases}$$

and the coefficients A_i, B_i are all arbitrary, unless determined by initial conditions.

144. The complete vibration therefore consists, as in the case of the unloaded string, of simple harmonic vibrations superposed. But the values of m, which determine the periods of these component vibrations, are not in general commensurable numbers, so that the component tones do not belong to a harmonic scale, and can only be improperly called 'harmonics.'

If in equation (34) we suppose $\lambda = 0$, or $b = 0$, or $b = l$, we get the condition for an unloaded string, namely, $\sin ml = 0$, as in the original investigation of that case.

If, on the other hand, we suppose $\mu = \infty$ (which we are at liberty to do if we also suppose gravity not to act), then $\lambda = \infty$, and the second member of (34) is insignificant in comparison with the first, and the condition for determining m becomes

$$\sin mb \sin m(l-b) = 0;$$

so that either $\sin mb = 0$, or $\sin m(l-b) = 0$.

Either of these equations gives $y = 0$ when $x = b$; that is, the point at which μ is attached remains fixed, as it evidently ought.

The periods of the component vibrations are now those which belong to the separate portions of the string; and equation (35) shews that each can only exist in its own portion. But it is evident that we may now, without violating any prescribed condition, take $\sin m b = 0$ in one portion and $\sin m (l - b) = 0$ in the other; thus the motion consists in general of the natural vibrations of the two portions, existing independently. The infinite attached mass is simply equivalent to a fixed point.

145. In general the roots of the transcendental equation (34) could only be found by troublesome approximations. Two special cases however deserve attention.

The first is that in which the mass μ is attached at a *node*, so that $b = \frac{j}{j'} l$, j, j' being integers. (34) then becomes

$$m \lambda \sin m b \sin \frac{j' - j}{j} m b = \sin \frac{j'}{j} m b;$$

and it is evident that this is satisfied by taking for mb any multiple of $j \pi$. Thus we shall get a series of roots (not *all* the roots) by giving i integer values from 1 to ∞ in

$$m_i = \frac{i j \pi}{b} = \frac{i j' \pi}{l}.$$

The period of the i^{th} component tone given by this series is, as we see from (35),

$$\frac{2 \pi}{a m_i} = \frac{2 l}{a i j'}.$$

Now $\frac{2 l}{a j'}$ is the period of the vibration of the unloaded string, when it is vibrating so as to give the lowest harmonic tone which has *nodes* at the points of division of the string into j' equal parts.

We see therefore that when the mass is attached at a node, all the component tones which have nodes at that point remain unaltered. But the fundamental, and other component tones, will be changed.

This may be verified by attaching a small lump of wax to one of the points of aliquot division of a violin or pianoforte string.

The other case is that in which the attached mass is so small that the square of the ratio $\frac{\lambda}{l}$ may be neglected.

Since (34) is satisfied by $ml = i \pi$ when $\lambda = 0$, we may assume that when λ is small it will be satisfied by $m\,l = i\pi + \epsilon$, where ϵ is of the same order as $\frac{\lambda}{l}$.

Substituting therefore $\frac{i\pi+\epsilon}{l}$ for m in (34), we obtain

$$\frac{\lambda}{l}(i\pi+\epsilon)\sin\frac{b}{l}(i\pi+\epsilon)\sin\left(1-\frac{b}{l}\right)(i\pi+\epsilon) = \sin(i\pi+\epsilon);$$

and, terms of the second order being neglected, this becomes

$$i\pi\frac{\lambda}{l}\sin i\pi\frac{b}{l}\sin\left(i\pi - i\pi\frac{b}{l}\right) = \epsilon\cos i\pi;$$

whence we get

$$-i\pi\frac{\lambda}{l}\sin^2 i\pi\frac{b}{l} = \epsilon;$$

so that we may take

$$m_i = \frac{i\pi+\epsilon}{l} = \frac{i\pi}{l}\left(1-\frac{\lambda}{l}\sin^2 i\pi\frac{b}{l}\right).$$

The period of the corresponding tone is $\frac{2\pi}{a\,m_i}$; or the number of vibrations in a unit of time is $\frac{a\,m_i}{2\pi}$; hence the number of vibrations is diminished by the load in the ratio of

$$1-\frac{\lambda}{l}\sin^2 i\pi\frac{b}{l} \text{ to } 1,$$

and the fundamental tone, as well as the higher components, are all lowered; moreover the components belong nearly but not exactly to a harmonic series, so that the compound note will sound slightly discordant. The examination of particular cases may be left to the reader.

NOTE.

On the Principles of the Use of Revolving Mirrors.

(See Art. 138.)

If a plane mirror revolve about a fixed axis AB in its own plane, the path of the image of any stationary point Q is a circle which passes through Q, and has its centre at the point where a perpendicular from Q meets AB. And this path is the same whether one side only, or each side, of the mirror be a reflecting surface.

If the axis of rotation, AB, be not in, but parallel to, the plane of the mirror, then the path of the image of Q is a curve of the 4th degree, having a double point at Q, and two loops, one within and the other without the circle described as above. The inner loop is the path of the image formed by reflection at the outer surface (reckoning from AB) of the mirror, and the outer loop of that formed by the inner surface.

A usual arrangement is to join four mirrors together so as to form four sides of a cubical box, with the axis of rotation passing through the centre of the box, parallel to their planes, and equidistant from them all. The outer surfaces of course alone reflect, and the images formed by them all describe the same path.

But an eye placed at any determinate point will only see one image at one time, and only while it describes a small portion of its path; and if the velocity of rotation be sufficiently great, this small portion of the path of the image of a stationary and continuously illuminated point will appear to the eye as a continuous and stationary line. If however the point, while continuously illuminated, have a vibratory motion of sufficiently short period, parallel to the axis of rotation, and if the velocity of rotation of the mirrors be so adjusted that one quarter of its period is equal to, or a multiple of, the period of vibration, then the passage of each mirror through any given position will always happen when the vibration is in the same phase; and consequently the visible portion of the path of the image will appear as one or more waves of a continuous and stationary vibration curve, formed by compounding the two motions along and perpendicular to the line before mentioned.

On the other hand, if the point be stationary, but illuminated only at instants separated by sufficiently short intervals of time, it will appear, when viewed by the eye directly, as a continuously illuminated point; but when seen by reflection from the revolving mirrors, it will appear, not as a continuous line, but as a row of points, which will be stationary if one quarter of the period of rotation of the mirrors be equal to, or a multiple of, the interval of time between successive illuminations.

Some of the most usual applications of revolving mirrors depend upon these principles. They appear to have been first used, for purposes of observation, by Wheatstone.

CHAPTER VIII.

ON THE LONGITUDINAL VIBRATIONS OF AN ELASTIC ROD.

146. THE vibrations of a uniform elastic rod may be either transversal or longitudinal, and both kinds may, when small, coexist without sensibly modifying each other. We may therefore study them separately; and we shall begin with the theory of longitudinal vibrations, as being the simplest. We might, as we did in the case of the string, first consider the subject kinematically, assuming the law of wave-propagation in a rod of infinite length; but we prefer to proceed at once to the dynamical theory.

147. We suppose then the motion of all the particles to be in directions parallel to a fixed straight line in space, with which the axis of the rod always coincides. By the *axis* of the rod is meant a line passing through the centres of inertia of its transverse sections.

Further, we suppose that all the particles which at any one instant are in a plane at right angles to the axis, continue to be so at all times. In other words, the velocities of all the particles in the same transverse section are equal.

The first of these suppositions cannot be rigorously true, because we know that a longitudinal extension of any part of the rod is in general accompanied by a lateral contraction, and *vice versâ*. But when the vibrations are small these lateral displacements may be neglected without sensible error.

148. We shall first investigate the conditions of equilibrium, and then deduce the equations of motion from them by the help of D'Alembert's principle.

The usual law of elasticity is assumed, namely,

$$T = q\,\epsilon,$$

where q is a constant (the modulus of elasticity), and T is the force, per unit of area, which must be applied, in contrary directions, to any two transverse sections, in order to produce an extension (or compression) ϵ. If T is a *tension*, or pulling force, the effect will of course be positive extension, or elongation; and if T be a pushing force (or negative tension) the effect will be negative extension, or contraction. The definition of extension, which includes both cases, is

$$\epsilon = \frac{\text{actual length}}{\text{natural length}} - 1.$$

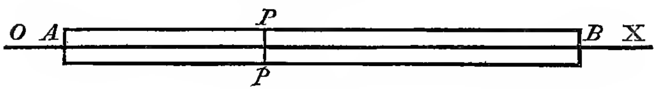

Fig. 1.

Let AB (Fig. 1) be the axis of the rod, coinciding with a line OX fixed in space. And let us suppose that the rod is actually in equilibro under the action of given forces, of which the directions are all parallel to OX. Let ξ be the *actual* abscissa of any transverse section Pp reckoned from the fixed origin O, and x the *natural* (or unextended) distance of the same section *from the end A.* (By the *same* section is meant the section containing the *same particles*.) And let ξ_0 be the value of ξ at A.

We have then to consider the conditions of equilibrium when the following external forces are applied:

(1) A force F_0 per unit of surface applied to the end A, and a force F_1 of the same kind at the end B. The usual rule of signs being adopted, F_0 will be a pushing force if it is positive, and a pulling force if it is negative, while the converse will be true of F_1.

(2) A force X per unit of mass applied throughout the infinitesimal slice between two sections of which the actual abscissæ are ξ and $\xi + d\xi$.

149. Let ω be the area of the section, and l the natural length of the rod. The equilibrium being established, if the part of the rod between Pp and B were cut off, in order to maintain the equilibrium of the remainder it would be necessary to apply to the surface of the section Pp some force F per unit of area, and the condition of equilibrium would be

$$F_0 \omega + \omega \int_{\xi_0}^{\xi} \rho X \, d\xi + F\omega = 0 ; \qquad (1)$$

where ρ represents the actual density in the section of which the abscissa is ξ, so that $\rho \omega X d\xi$ is the whole force upon the interior mass of the slice.

To find an expression for F we observe, that the natural thickness of any slice being dx, and the actual thickness $d\xi$, the extension is $\dfrac{d\xi}{dx} - 1$, and the force per unit of area, on each face, required to produce this extension, is therefore

$$q\omega \left(\frac{d\xi}{dx} - 1 \right),$$

supposing no forces (such as X) to act on the interior mass of the slice. Now when the thickness of the slice is diminished without limit, the forces on its faces remain finite, being proportional to *areas*, whereas the interior forces, if there are any, being proportional to *volume*, diminish without limit also, and are therefore negligeable in comparison with the forces on the faces. Hence $q\omega \left(\dfrac{d\xi}{dx} - 1 \right)$ is the force which must be applied to the surface of the section to maintain the existing state of extension; this therefore is the value of F. Thus (1) becomes

$$F_0 + \int_{\xi_0}^{\xi} \rho X \, d\xi + q \left(\frac{d\xi}{dx} - 1 \right) = 0 ;$$

which, since $x = 0$ when $\xi = \xi_0$, may be written

$$F_0 + \int_0^x \rho X \frac{d\xi}{dx} dx + q \left(\frac{d\xi}{dx} - 1 \right) = 0. \qquad (2)$$

Differentiating this equation with respect to x, we obtain

$$\rho X \frac{d\xi}{dx} + q \frac{d^2\xi}{dx^2} = 0.$$

Let ρ_0 be the natural density of the rod; then, since $\rho_0\, \omega\, dx$ and $\rho\, \omega\, d\xi$ both express the mass of the same slice, we have

$$\rho \frac{d\xi}{dx} = \rho_0,$$

so that the last equation becomes

$$X + \frac{q}{\rho_0} \frac{d^2\xi}{dx^2} = 0. \tag{3}$$

150. To deduce from this the equation of motion, in the case in which no forces are actually applied except on the surfaces of the ends, we have merely to substitute for the force $\omega\, \rho_0 X\, dx$ supposed to act on the slice, the resistance to acceleration arising from its inertia, namely, $-\omega\, \rho_0\, dx \,.\, \frac{d^2\xi}{dt^2}$; that is, to write $-\frac{d^2\xi}{dt^2}$ instead of X. Thus we obtain, putting $\frac{q}{\rho_0} = a^2$,

$$\frac{d^2\xi}{dt^2} = a^2 \frac{d^2\xi}{dx^2}; \tag{4}$$

an equation of the same form as that to which we were led by the problem of vibrating strings.

The solution of this equation gives ξ as a function of the two independent variables x and t. That is, it gives the position, at any time t, of any proposed section defined by the value of x, its natural distance from the end A.

The solution is

$$\xi = \phi\,(x - at) + f\,(x + at); \tag{5}$$

and the two arbitrary functions have to be determined by the initial displacements and velocities, together with the given conditions relative to the extremities.

151. If in equation (2) we put for $\rho \frac{d\xi}{dx}$ its value ρ_0, we get

$$F_0 + \rho_0 \int_0^x X\, dx + q\left(\frac{d\xi}{dx} - 1\right) = 0;$$

and, putting $x = 0$ in this,

$$F_0 = -q\left(\left(\frac{d\xi}{dx}\right)_{x=0} - 1\right). \tag{6}$$

If we had considered the equilibrium of the other part of the rod, we should have found in like manner

$$F_1 = q\left(\left(\frac{d\xi}{dx}\right)_{x=l} - 1\right). \tag{7}$$

These two equations merely express that the condition of extension at each end of the rod is always such as corresponds to the force applied there.

152. The differential equation (4) is satisfied by such a value as

$$\xi = A \sin(mx + \alpha) \sin(mat + \beta);$$

but in order to satisfy the given conditions in all cases, we shall find it necessary to add a non-periodic term such as

$$b + c(x - at) + c'(x + at),$$

which is obviously of the general form (5) and satisfies (4). But the part $b + (c' - c)at$ of this term merely signifies a uniform motion of translation of the whole rod. Such a motion may, if the terminal conditions admit it, exist; but as it in no way modifies the relative motion of the different sections, which is all we are concerned with in studying the vibrations, we may neglect it, and assume only a term kx in addition to the periodic part of ξ.

153. We proceed to consider the most important cases. And first we will suppose the rod entirely free, and acted on by no forces. Then $F_0 = 0$, $F_1 = 0$, and the two equations (6), (7), give

$$\frac{d\xi}{dx} = 1,$$

both when $x = 0$ and when $x = l$, as the terminal conditions.

Assuming then

$$\xi = kx + A \sin(mx + \alpha) \sin(mat + \beta),$$

the terminal conditions are

$$k + mA \cos(mx + \alpha) \sin(mat + \beta) = 1,$$

when $x = 0$ and when $x = l$, for all values of t.

Hence we must evidently have

$$k = 1, \qquad \cos\alpha = 0, \qquad \cos(ml + \alpha) = 0;$$

of which equations the last two are satisfied by $\alpha = \frac{\pi}{2}$, $ml = i\pi$, (i being any integer), and therefore

$$\xi = x + A\cos\frac{i\pi x}{l}\sin\left(\frac{i\pi a t}{l} + \beta\right)$$

is a solution, A and B being arbitrary; and making a slight change of form, as in former cases, we may take

$$\xi = x + \sum_{i=1}^{i=\infty}\cos\frac{i\pi x}{l}\left(A_i\cos\frac{i\pi a t}{l} + B_i\sin\frac{i\pi a t}{l}\right) \quad (8)$$

as the general solution.

(If we included $i = 0$ in the summation, we should merely add a constant term to ξ, which would be equivalent to an alteration of the fixed origin from which ξ is measured).

154. To understand the equation (8) we must recollect that ξ is the distance from a fixed origin, at the time t, of the particles in a plane section of which the natural distance from the end A is x. The value of x therefore depends only on the particular set of particles considered, and is independent of the origin of ξ.

Let us now find the position of the centre of inertia of the rod. Its abscissa $\bar{\xi}$ is given by the equation

$$\bar{\xi}\int\rho\, d\xi = \int\rho\,\xi\, d\xi,$$

the integrations being extended from one end to the other. Now we have (Art. **149**),

$$\rho = \rho_0\left(\frac{d\xi}{dx}\right)^{-1};$$

hence if we put, in the above equation, $\frac{d\xi}{dx}dx$ for $d\xi$, so that the limits of the integrations are $x = 0$ and $x = l$, it becomes simply

$$\bar{\xi}\, l = \int_0^l \xi\, dx;$$

but from (8) we have $\int_0^l \xi\, dx = \frac{l^2}{2}$; hence

$$\bar{\xi} = \frac{l}{2},$$

or the centre of inertia remains fixed (as we know *à priori* that it must do under the supposed conditions), and is at a distance $\frac{l}{2}$ from the origin of ξ. It must not however be inferred from this that the section which, in the natural condition, contains the centre of inertia remains fixed; that will only happen when A_i, B_i are 0 for all even values of i. In general, *no* section remains fixed, the place of the centre of inertia being occupied by different particles periodically.

155. If the vibrations ceased, the centre of inertia still maintaining its position, the periodic part of (8) would disappear, and we should have $\xi = x$ at all points of the rod; hence the periodic part, which is the actual value of $\xi - x$, gives the displacement, at the time t, of the section defined, as before explained, by the value of x.

The density at any point is given (Art. **149**) by the equation

$$\frac{1}{\rho} = \frac{1}{\rho_0} \frac{d\xi}{dx}.$$

Hence the general equation (5) gives

$$\frac{\rho_0}{\rho} = \phi'(x - at) + f'(x + at);$$

which represents the transmission of two *states of density* in contrary directions with the same constant velocity a relatively to the matter of the rod. This is not rigorously the same thing as a constant velocity relatively to fixed space, because x, in the state of motion, is not the actual abscissa of the particles in a given section, reckoned from a fixed point, but differs from it by the small periodic displacement due to the vibration.

This being understood, we may say that the vibration consists in the transmission, in contrary directions, of waves of condensation and dilatation; just as the lateral vibration of a string consists in the transmission of waves of lateral displacement; and the waves appear to be reflected from the ends in both cases.

156. The periodic part of (8) does not in general vanish for any value of x, so that there are in general no *nodes*, or sections of *no displacement*. But there will be n nodes, at sections for which x is any odd multiple of $\frac{l}{2n}$, provided A_i, B_i vanish for all values of i except odd multiples of n. Thus the rod *may* have any number of nodes, of which those next the ends are distant from the ends by half the distance between any two nodes.

From (8) we have also $\frac{\rho_0}{\rho} = \frac{d\xi}{dx}$

$$= 1 - \frac{\pi}{l} \Sigma i \sin \frac{i\pi x}{l} \left(A_i \cos \frac{i\pi a t}{l} + B_i \sin \frac{i\pi a t}{l}\right);$$

hence $\frac{\rho_0}{\rho} = 1$ when $x = 0$ and when $x = l$. That is, there is no variation of density at the free ends. But there will in general be variation of density at all other points. If A_i, B_i vanish except when i is a multiple of n, the variable part of $\frac{d\xi}{dx}$ vanishes when x is a multiple of $\frac{l}{n}$. Hence, when there are nodes, the sections in which there is no variation of density are those which bisect the nodal intervals in the state of equilibrium, and these sections of no variation of density are also sections of greatest displacement, as will be seen on inspection of (8), since $\cos \frac{i\pi x}{l}$ is ± 1 for values of x which make $\sin \frac{i\pi x}{l} = 0$.

157. The vibration represented by (8) consists as usual of the superposition of an infinite number of simple harmonic vibrations, each of which might subsist by itself; the i^{th} component vibration would have i nodes; and in this case, as in the case of the string, the tones corresponding to the component vibrations form in general a complete harmonic series. The period of the i^{th} component tone is $\frac{2l}{ia}$, and the period of the fundamental tone is $\frac{2l}{a}$.

Since $\frac{2l}{a}$ would be the time of transmission of a wave over the distance $2l$, we infer, exactly as in the case of a string, that the wave-length is twice the length of the rod. Since the value of a (Art. **150**) is $\left(\frac{q}{\rho_0}\right)^{\frac{1}{2}}$, the number of vibrations in a unit of time, or $\frac{a}{2l}$, is

$$\frac{1}{2l}\left(\frac{q}{\rho_0}\right)^{\frac{1}{2}};$$

and is therefore, *cæteris paribus*, inversely proportional to the length. It is, as it evidently ought to be, independent of the thickness.

158. The most general case in which there is a node at the middle of the rod is that in which $\cos\frac{i\pi x}{l}$ vanishes, for all values of i included in the series (8), when $x=\frac{l}{2}$. In order that this may be the case, A_i, B_i must vanish for all even values of i. The gravest component tone is then the fundamental tone of the rod, but the higher tones of even orders disappear. Thus the first upper tone will be at an interval of a *twelfth* (octave + fifth) above the fundamental tone.

Now in this case the middle section of the rod might become absolutely fixed without disturbing the motion, and either half might then be taken away, so as to leave a rod of half the original length, with one end fixed and the other free.

Hence we infer that the fundamental tone of a rod, with one end fixed, is the same as that of a rod of twice the length, with both ends free. But the component tones of the rod with a fixed end do not form a complete harmonic series, containing only the tones of odd orders. The wave-length is four times the length of the rod. We shall arrive at the same conclusions afterwards in a more direct manner.

159. We will next suppose the terminal sections of both ends to be fixed.

Let l' be the distance between the planes of the fixed ends. Then, if l' be different from the natural length l, the rod, when at rest, is in a state of uniform elongation or contraction, maintained by the tensions or pressures on the fixed ends. We will suppose, for clearness, that $l' > l$, so that the rod is elongated.

Assuming, as in Art. **153**,

$$\xi = kx + A\sin(mx+a)\sin(mat+\beta)$$

(in which equation x has its original meaning), we have to satisfy the conditions $\frac{d\xi}{dt}=0$ when $x=0$ and when $x=l$, for all values of t. Hence we must have $\sin(mx+\alpha)=0$ when $x=0$ and when $x=l$, which conditions are satisfied by taking

$$\alpha=0, \qquad ml=i\pi;$$

moreover, since the ends are now fixed, we may assume the origin from which ξ is measured to coincide with one end, so that $\xi=0$ when $x=0$; then k must be such that $\xi=l'$ when $x=l$, or $k=\frac{l'}{l}$. Thus the expression above assumed for ξ becomes

$$\xi=\frac{l'}{l}x+A\sin\frac{i\pi x}{l}\sin\left(\frac{i\pi a t}{l}+\beta\right); \tag{9}$$

now let x' be the distance of the section defined by x from the end of the rod when at rest in its actual condition of extension, then $x'=\frac{l'}{l}x$; and if we put $a'=\frac{l'}{l}a$, and take, as before, the sum of the particular solutions for the general solution, we obtain

$$\xi=x'+\Sigma\sin\frac{i\pi x'}{l'}\left(A_i\cos\frac{i\pi a' t}{l'}+B_i\sin\frac{i\pi a' t}{l'}\right). \tag{9'}$$

This equation evidently expresses a vibration in which the velocity of wave-transmission is $a'=\frac{l'}{l}a$[1]. Thus the tension to which we have supposed the rod subjected increases the velocity

[1] This value of the velocity of wave-transmission might be obtained directly thus: let T' be the tension in the state of rest, T the actual tension at any point; then

$$T'=q\left(\frac{dx'}{dx}-1\right) \quad \text{and} \quad T=q\left(\frac{d\xi}{dx}-1\right);$$

from which we find by eliminating dx,

$$T=T'+(q+T')\left(\frac{d\xi}{dx'}-1\right);$$

if we now investigated directly the differential equation, we should find

$$\frac{d^2\xi}{dt^2}=a'^2\frac{d^2\xi}{dx'^2} \quad \text{where} \quad a'^2=\frac{q+T'}{\rho'},$$

ρ' being the density of the rod at rest. But ρ_0 being the unextended density, we have $\frac{\rho_0}{\rho'}=\frac{l'}{l}=1+\frac{T'}{q}$;

$$\text{hence} \qquad a'^2=q\frac{\rho_0}{\rho'^2}=\left(\frac{l'}{l}\right)^2\frac{q}{\rho_0}=\left(\frac{l'}{l}\right)^2a^2.$$

of transmission in proportion to the extension. But the period $\frac{2l'}{a'} = \frac{2l}{a}$, is the same as if there was no tension.

160. Comparing (9′) with (8), we see that the periods of the fundamental and other component tones are the same in the rod with both ends fixed as in that with both ends free. But when there are nodes they are not at the same places. The rod with fixed ends has always two nodes, namely, the fixed ends themselves; the ith harmonic component would have $i-1$ nodes (besides the ends) dividing the rod into i equal parts. The mode of division in the free rod was explained in Art. **156**.

161. The theory of the longitudinal vibrations of a rod extended by tension at its ends, is evidently applicable at once to the case of a string similarly extended, in so far as the assumed relation between tension and extension may be supposed to subsist.

The longitudinal vibrations of a pianoforte string may be excited by gently rubbing it longitudinally with a piece of india-rubber, and those of a violin string by placing the bow obliquely across the string, and moving it along the string longitudinally, keeping the same point of the bow upon the string. The note is unpleasantly shrill in both cases. (The relation between the pitch of lateral and longitudinal vibration will be considered afterwards.)

If the peg of the violin be turned so as to alter the pitch of the lateral vibrations very considerably, it will be found that the pitch of the longitudinal vibrations has varied very slightly. The reason of this is that in the case of the lateral vibrations the change of velocity of wave-transmission depends chiefly on the change of tension, which is considerable. But in the case of the longitudinal vibrations, the change of velocity of wave-transmission depends on the change of extension, which is comparatively slight. For experiments on the longitudinal vibrations of rods, it is convenient to use rods of deal, or of steel, or glass tubes. One end may be fixed in a stand; or the rod may be held lightly in the fingers at the place of a node.

The vibrations may be excited by rubbing the glass with a wet cloth, and the rods with powdered rosin on a dry glove.

162. If, instead of supposing both ends of the rod fixed, we supposed a *constant force* applied at each end, the terminal conditions would be $F_0 = -F_1 =$ constant. Hence (Art. **151**) $\frac{d\xi}{dx} - 1$ must have the same given value, say $e-1$, at both ends. Assuming then, as before (Art. **153**),

$$\xi = kx + A\sin(mx+\alpha)\sin(mat+\beta),$$

we must have

$$k - 1 + mA\cos(mx+\alpha)\sin(mat+\beta) = e - 1$$

both when $x = 0$ and when $x = l$, and therefore

$$k = e, \qquad \cos\alpha = 0, \qquad \cos(ml+\alpha) = 0;$$

whence, as in Art. **153**,

$$\alpha = \frac{\pi}{2}, \qquad ml = i\pi;$$

$$\xi = ex + \sum_{i=1}^{i=\infty} \cos\frac{i\pi x}{l}\left(A_i\cos\frac{i\pi at}{l} + B_i\sin\frac{i\pi at}{l}\right);$$

and, putting $ex = x'$, $el = l'$, $ea = a'$, as in Art. **159**, we should have

$$\xi = x' + \sum_{i=1}^{i=\infty} \cos\frac{i\pi x'}{l'}\left(A_i\cos\frac{i\pi a't}{l'} + B_i\sin\frac{i\pi a't}{l'}\right);$$

in which x' now signifies the distance of any section from the end A of the rod, supposing it to be at rest under the action of the terminal forces.

Comparing this result with equation (8) we see that the period of the i^{th} component vibration $\left(= \frac{2l'}{a'} = \frac{2l}{a}\right)$ is the same in both cases, and the nodes are similarly situated.

We infer then that in the three cases, (1) both ends free, (2) both ends fixed, (3) both ends pulled or pushed by equal constant forces, the series of component tones is the same. But the distribution of the nodes, which is the same in (1) and (3), is different in (2).

The case (3) cannot be realized in practice, because it is impossible by any mechanical contrivance to apply a constant force at the ends. In case (2) the force supplied by the fixed supports of the ends is a periodically varying quantity, of which the actual value is that of $-q\left(\frac{d\xi}{dx}-1\right)$ at the end A, and $q\left(\frac{d\xi}{dx}-1\right)$ at the other end, and the *mean* value is

$$\mp q\left(\frac{l'}{l}-1\right).$$

163. The only remaining case of practical interest is that in which one end of the rod is fixed, while the other end is either entirely free, or loaded with a given finite mass.

We shall shew how the solution of the problem in these cases may be obtained by means of the more general supposition that both ends of the rod are loaded, but otherwise free. The condition of fixity at either end can then be introduced by supposing the mass attached at that end to be infinite, and the condition of perfect freedom by supposing it to be nothing.

Suppose then masses M_0, M_1 to be attached to the two ends. The forces F_0, F_1 (Art. **151**) will then be the resistances to acceleration arising from the inertia of these masses; and the terminal conditions will therefore be

$$\omega F_0 = -M_0\left(\frac{d^2\xi}{dt^2}\right)_{x=0}, \qquad \omega F_1 = -M_1\left(\frac{d^2\xi}{dt^2}\right)_{x=l};$$

whence (see equations (6), (7)),

$$\frac{d\xi}{dx} - 1 = \frac{M_0}{q\omega}\frac{d^2\xi}{dt^2} \quad \text{when} \quad x = 0,$$

$$\frac{d\xi}{dx} - 1 = -\frac{M_1}{q\omega}\frac{d^2\xi}{dt^2} \quad \text{when} \quad x = l.$$

And if in these equations we substitute the values of $\frac{d\xi}{dx}$, $\frac{d^2\xi}{dt^2}$, from the assumed equation

$$\xi = kx + A\sin(mx+\alpha)\sin(mat+\beta), \tag{10}$$

we see that, in order to satisfy them for all values of t, we must have $k=1$ and

$$\cot(mx+\alpha) = -ma^2\frac{M_0}{q\omega} \quad \text{when} \quad x = 0,$$

$$= \quad ma^2\frac{M_1}{q\omega} \quad \text{when} \quad x = l.$$

Let $\frac{a^2M_0}{lq\omega} = \mu_0$, $\frac{a^2M_1}{lq\omega} = \mu_1$, then these conditions give

$$\left.\begin{aligned} \cot\alpha &= -\mu_0 ml, \\ \cot(ml+\alpha) &= \mu_1 ml; \end{aligned}\right\} \tag{11}$$

from which, by eliminating a, we find

$$(1-\mu_0\mu_1(ml)^2)\tan ml + (\mu_0+\mu_1)\,ml = 0. \qquad (12)$$

164. Suppose m_1, m_2, &c. are the values of m which satisfy the equation (12). To each value m_i will correspond a value of a, say a_i, which can be found from (11). Then (10) will give the form

$$\xi = x + \Sigma \sin(m_i x + a_i)(A_i \cos m_i a t + B_i \sin m_i a t)\,; \qquad (13)$$

where A_i, B_i are arbitrary constants.

This would give the solution of the problem if the values of m_i were known. These values in general could only be found by troublesome approximations. We see however that (13) expresses a vibration compounded of simple harmonic vibrations, of which the periods are inversely proportional to the values of m_i, so that the component tones do not in general belong to a harmonic scale.

165. If we suppose $\mu_0 = 0$ and $\mu_1 = 0$, then both ends of the rod are perfectly free, and equations (11) give

$$\cot a = 0, \qquad \cot(ml + a) = 0\,;$$

or $a = \frac{\pi}{2}$, $ml = i\pi$, as we found before.

Again, if $\mu_0 = \infty$ and $\mu_1 = \infty$, then both ends are fixed, and we have

$$\cot a = \infty, \qquad \cot(ml + a) = \infty,$$

$$\text{or} \quad a = 0, \qquad ml = i\pi,$$

as we also found before in the case of fixed ends without extension.

But if $\mu_0 = \infty$, $\mu_1 = 0$, then the end A is fixed and the other end free; and we have from (11),

$$\cot a = \infty, \qquad \cot(ml + a) = 0,$$

$$\text{or} \quad a = 0, \qquad ml = (2i+1)\frac{\pi}{2},$$

so that (13) becomes

$$\xi = x + \Sigma \frac{(2i+1)\pi x}{2l}$$

$$\times\left(A_i \cos \frac{(2i+1)\pi a t}{2l} + B_i \sin \frac{(2i+1)\pi a t}{2l}\right). \qquad (14)$$

Here the periods of the component vibrations are the values of $\frac{4l}{(2i+1)a}$, and the numbers of vibrations in a unit of time are therefore proportional to the odd numbers 1, 3, 5, ... Thus the component tones form a harmonic scale with alternate tones (namely, the octave of the fundamental tone with all its harmonics) left out. The wave-length of the fundamental tone is $4l$, and its pitch is the same as that of a rod of length $2l$, fixed at both ends or free at both. (See Art. **158**.)

166. Lastly, we shall consider the case of a rod fixed at one end, and loaded with a small mass at the other. Then we shall have

$$\mu_0 = \infty, \qquad \mu_1 = \epsilon \quad \text{(a small quantity)};$$

and therefore, from (11),

$$\cot a = \infty, \qquad \cot(ml + a) = \epsilon m l:$$

the first of these is satisfied by $a = 0$, and the second becomes

$$\cos ml = \epsilon ml \sin ml.$$

Now if ϵ were $= 0$ the solution of this would be $ml = (2i+1)\frac{\pi}{2}$; we may therefore assume

$$m_i l = (2i+1)\frac{\pi}{2} + \theta,$$

θ being a small quantity of the same order as ϵ. Then

$$\cos\left((2i+1)\frac{\pi}{2} + \theta\right) = \epsilon\left((2i+1)\frac{\pi}{2} + \theta\right)\sin\left((2i+1)\frac{\pi}{2} + \theta\right);$$

or, quantities of the second order being neglected,

$$-\theta \sin(2i+1)\frac{\pi}{2} = \epsilon(2i+1)\frac{\pi}{2} \cdot \sin(2i+1)\frac{\pi}{2};$$

whence $$\theta = -\epsilon(2i+1)\frac{\pi}{2},$$

and $$m_i l = (2i+1)\frac{\pi}{2}(1-\epsilon).$$

This shews that the effect of the small load is simply to diminish the number of vibrations of every component tone in the ratio $1-\epsilon : 1$. Each tone is therefore lowered by the same interval, and the whole series still belongs to a harmonic scale with alternate tones omitted[2].

[2] The following result is found by carrying the approximation one step further.

Let ϵ mean the ratio of the attached mass to the whole mass of the rod *and* the attached mass. Then

$$n_i' = n_i\left\{1 - \epsilon + \frac{(2i+1)^2}{12}\pi^2\epsilon^3\right\}$$

where n_i is the number of vibrations, in a unit of time, of the unloaded rod.

167. Since $a^2 = \dfrac{q}{\rho_0}$ (Art. **150**), the values of μ_0, μ_1 (Art. **163**) are

$$\mu_0 = \frac{M_0}{l\,\rho_0\,\omega}, \qquad \mu_1 = \frac{M_1}{l\,\rho_0\,\omega};$$

now $l\rho_0\omega$ is the mass of the rod; hence μ_0, μ_1 are simply the ratios of the attached masses to the mass of the rod, and ϵ in the last problem has the same meaning.

168. The results of this chapter afford a method of determining experimentally the modulus of elasticity by observing the tones produced by longitudinal vibrations. Thus, taking the case of the rod with one end fixed and the other free, we have for the period of the fundamental tone (Art. **165**),

$$\frac{4l}{a} = 4l.\left(\frac{\rho_0}{q}\right)^{\frac{1}{2}},$$

and therefore if n be the number of vibrations in a unit of time,

$$16\,n^2 l^2 = \frac{q}{\rho_0} \quad \text{and} \quad q\,\omega = 16\,n^2 l\,.\,l\rho_0\,\omega;$$

now if a second be the unit of time, the weight of the rod expressed in theoretical units of force is $l\rho_0\,\omega g$; hence, calling this W, we have

$$q\,\omega = \frac{16\,n^2 l}{g}\,.\,W.$$

Thus the pulling force which would double the length of the rod, if the law of extension held good without limit, is

$$\frac{16\,n^2 l}{g} \times \text{weight of rod.}$$

The value of n can be ascertained with great accuracy by methods of which the principle will be explained afterwards.

Hence the higher component tones become *unharmonic*. For instance, the ratio of the interval between the fundamental tone and the first upper tone is (to the same approximation) $3\,(1 + \frac{4}{3}\pi^2\epsilon^3)$, which exceeds a twelfth by an interval of which the ratio is $1 + \frac{4}{3}\pi^2\epsilon^3$. This would be a diatonic semitone if $\frac{4}{3}\pi^2\epsilon^3 = \frac{1}{15}$, or $\epsilon^3 = \dfrac{1}{20\,\pi^2}$, which gives $\frac{5}{24}$ nearly for the ratio of the attached mass to the mass of the rod.

The interval by which the first upper tone is put out of tune relatively to the fundamental tone, being measured by the logarithm of $1 + \frac{4}{3}\pi^2\epsilon^3$, varies as ϵ^3 nearly.

CHAPTER IX.

ON THE LATERAL VIBRATIONS OF A THIN ELASTIC ROD.

169. THE theory of the lateral vibrations of a rod becomes susceptible of tolerably simple mathematical treatment when the following assumptions are made.

The rod is supposed to be homogeneous. In its undisturbed condition it is straight, and all its transverse sections are similar, equal, and similarly situated.

A line passing through the centres of inertia of all transverse sections may be called the *axis of the rod.*

We suppose the vibrations to be small, and such that

(1) One principal axis of every section remains in a fixed plane.

(2) No part of the axis of the rod undergoes any elongation or contraction.

(3) The particles which in the undisturbed state are in any transverse plane section, remain always in a plane normal to the axis of the rod.

(4) The principal axis mentioned in (1) is *small.* (This is what is meant by calling the rod thin.)

The plane which always contains the principal axis mentioned in (1) may be called the *plane of vibration.*

It follows evidently from the above assumptions that the condition of the whole rod at any time is determined by the position and form of its axis.

170. Taking rectangular coordinate axes fixed in space, we will suppose, for clearness, that the axis of x is directed horizontally from left to right, and the axis of y vertically upwards. Thus the plane of zx is horizontal. We will also suppose that the axis of the rod in its undisturbed condition coincides with the x-axis, and that the plane of vibration is vertical, coinciding with the plane of xy. Thus the *small* principal axis of every section is always in the vertical plane of xy, and the other (which may or may not be small) is always horizontal. Calling the axis of the rod AB, we will suppose that in the undisturbed condition the left-hand end A coincides with the origin.

Let x be the abscissa of any given particle in the axis of the rod in the undisturbed condition. We suppose (as in the case of the string) that the vertical displacement of any particle in the axis is so small that its horizontal displacement may be neglected. Hence we may consider that x remains constant for a given particle in the axis, and is the same as the abscissa of that particle reckoned from the fixed origin. Then, if y be the (vertical) ordinate of the same particle, y is always a small quantity. Also, if ds be an element of length of the axis, we may put $\frac{dx}{ds} = 1$, $\frac{dy}{ds} = \frac{dy}{dx}$, as in the case of the string. The problem is to express y as a function of the two independent variables x and t.

171. In the undisturbed condition let ρ be the density of the rod, ω the area of the section, and l the actual length of the axis (which remains constant).

If either one or both ends of the axis are free, l is the natural length of the rod. But if both ends are fixed at a distance different from the natural length, the axis is in a state of permanent extension or contraction, and l is not the natural length.

When either end of the axis is fixed, the whole of the terminal face at that end may or may not be fixed. When both ends of the axis are so fixed that there is extension or contraction, then if either terminal face be not entirely fixed, we must suppose normal tensions or pressures applied at all points of its surface such as would, in the undisturbed condition, maintain all the longitudinal filaments of the rod at the same length as the axis.

172. We must first investigate the conditions of equilibrium of the rod under the action of such forces as could produce a displacement of the kind supposed to exist at any time during the motion.

The usual rule of signs will be observed with respect to forces; and *moments* will be considered positive which tend to produce rotation *from* the axis of x towards that of y.

The slice included between two plane sections cutting the x-axis at distances x, $x+dx$ from the origin, always contains the same matter, though its faces are not in general parallel in the disturbed condition. We shall suppose

(1) A vertical force Y per unit of mass, constant throughout the same slice (so that Y is a function of x).

(2) Forces parallel to the plane of vibration, acting on the particles of the slice, and reducible to a couple, in that plane, of which the moment is

$$L \times (\text{mass of slice}) = L\,\rho\,\omega\,dx,$$

L being a function of x. (If there were such forces *not* reducible to a couple, they would in general tend to produce extension or contraction of the axis.)

(3) A force T_0 per unit of area, applied at every part of the terminal face A, at right angles to its plane.

(4) Tangential forces in the plane of the same face, parallel to the plane of vibration, and reducible to a single force $F_0\,\omega$, in that plane, applied at its centre.

(5) Forces applied to the surface of the face A, reducible to a couple in the plane of vibration, of which the moment is $G_0\,\omega$.

(6) Analogous forces applied at the face B, and denoted by T_1, F_1, G_1.

In the condition of equilibrium these forces are balanced by the forces of elasticity called into action by the state of strain which they produce in the rod.

Suppose then the equilibrium to subsist. It would not be disturbed if the part of the rod included between any two transverse sections became rigid. And if the rest of the rod were then removed, in order to maintain the equilibrium of this part it would be necessary to apply to it certain forces. We proceed to ascertain what these forces are, on the supposition that the part we have supposed to become rigid is an infinitesimal slice, and to deduce the differential equation which expresses the condition of equilibrium.

173. In Fig. 1 suppose the plane of the paper to be the plane of vibration, and let ab be an infinitesimal portion of the axis of the rod, and FG the section of an infinitesimal slice contained between transverse sections cutting the axis in a and b.

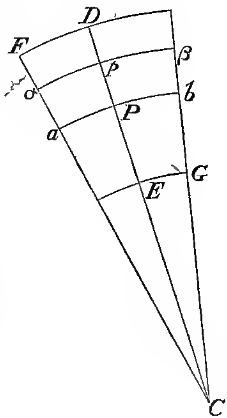

Fig. 1.

Let x be the abscissa of P, the middle point of ab, and $x - \frac{1}{2}dx$, $x + \frac{1}{2}dx$ the abscissae of a, b.

Suppose the transverse sections through a, b meet in C; then (quantities of the third order being neglected) PC is the radius of curvature of the axis at P, which we will call R.

Suppose the slice to be made up of longitudinal filaments having $d\omega$ for the area of their section; and let $\alpha p \beta$ be the projection of such a filament on the plane of vibration. Then, if $Pp = \eta$, it is plain that

$$\alpha\beta = \frac{pC}{PC} \cdot ab = \left(1 + \frac{\eta}{R}\right) dx.$$

Now the state of extension of any such filament may (since its length is infinitesimal) be considered to vary uniformly from one end to the other, so that we may obtain the state of extension at its middle point by calculating it as if it had the same value at all its points, that is, by the formula

$$\frac{\text{actual length}}{\text{natural length}} - 1.$$

Let then dx_0 be the natural length of dx or ab. This is also the natural length of $\alpha\beta$, so that the extension at p is

$$\left(1 + \frac{\eta}{R}\right)\frac{dx}{dx_0} - 1.$$

Now $\dfrac{dx}{dx_0} - 1$ is the extension of the axis (which is constant), so that if we put

$$q\left(\frac{dx}{dx_0} - 1\right) = T,$$

where q is the modulus of elasticity, T will be the value of a constant tension, per unit of sectional area, due to the permanent extension.

Then, calling T' the actual tension in the filament $\alpha\beta$, we have

$$T' = q\left(\left(1 + \frac{\eta}{R}\right)\frac{dx}{dx_0} - 1\right) = T + \frac{\eta}{R}(q + T).$$

Hence the pulling force exercised by the filament upon any element $d\omega$ of the section DE (on either side of it) is

$$\left(T + \frac{\eta}{R}(q + T)\right)d\omega.$$

Consider all these forces on one side (say the left) of the section. They are reducible to a resultant force $= -\int T'\,d\omega$ applied at P, and a couple of which the moment is $\int \eta\, T' d\omega$, the integrations being extended over the whole section. Now since the ordinate η is reckoned from a horizontal axis in the plane of the section, passing through its centre of inertia P, we have

$$\int \eta\, d\omega = 0, \quad \text{and} \quad \int \eta^2\, d\omega = \omega\kappa^2,$$

where κ is the radius of gyration of the area of the section about the horizontal principal axis in its plane. Thus the forces acting on the left-hand side of the section, due to extension, are equivalent to a resultant force $-T\omega$ perpendicular to its plane, applied at P, and a couple in the plane FG of which the moment is $\dfrac{q+T}{R}\kappa^2\omega$.

174. We can now find the forces which must be applied to the elementary slice in order to maintain its equilibrium when the rest of the rod is supposed to be removed, the slice having become rigid.

If we call Q the moment (which we have just determined) of the couple due to extension acting on the left-hand side of the section DE, then the moment of that on the left-hand face of the slice will be

$$Q - \tfrac{1}{2}\frac{dQ}{dx}\,dx,$$

and of that on the right-hand face

$$-\left(Q + \tfrac{1}{2}\frac{dQ}{dx}\,dx\right).$$

The sum of these is

$$-\frac{dQ}{dx}\,dx = -(q+T)\,\kappa^2\omega\,\frac{d}{dx}\left(\frac{1}{R}\right)dx. \qquad (a)$$

Moreover, we found a resultant pulling force $T\omega$ on each side of P. Hence on each face of the slice there is a resultant pulling force, applied perpendicularly at a and b respectively. The horizontal components of these forces may be considered equal (in opposite directions). But the vertical components $\left(\text{since } \frac{dy}{ds} = \frac{dy}{dx}\right)$ are

$$-T\omega\left(\frac{dy}{dx} - \tfrac{1}{2}\frac{d^2y}{dx^2}\,dx\right) \quad \text{at } a;$$

$$T\omega\left(\frac{dy}{dx} + \tfrac{1}{2}\frac{d^2y}{dx^2}\,dx\right) \quad \text{at } b;$$

these are equivalent to

$$\text{a vertical resultant force } = T\omega\,\frac{d^2y}{dx^2}\,dx, \qquad (b)$$

and *no* couple. (For the moment of the couple resulting from the horizontal components is

$$-T\omega\,dy, \qquad = -T\omega\,\frac{dy}{dx}\,dx,$$

while that from the vertical components is

(difference of forces) $\times \frac{1}{2}$ (distance between them) $= T\omega\,\frac{dy}{dx}\,dx$).

But we must not assume that the only forces lost by the removal of the other parts of the rod are the couple and resultant force just found; for the parts removed will in general have exercised *tangential* forces in the planes of the faces of the slice, reducible to resultant forces in the plane of vibration and applied at the centres of the faces.

Suppose then $\pm F\omega$ is the value of this tangential resultant force on the left and right sides of the section DE; then, on the left-hand face of the slice, it will be

$$\omega\left(F - \tfrac{1}{2}\frac{dF}{dx}dx\right),$$

and on the right-hand

$$-\omega\left(F + \tfrac{1}{2}\frac{dF}{dx}dx\right).$$

These expressions will also give $\left(\text{as we neglect } \left(\frac{dy}{dx}\right)^2\right)$ the vertical components; and therefore the forces in question are equivalent to

$$\text{a resultant vertical force } -\omega\frac{dF}{dx}dx, \qquad (c)$$

and a couple of which the moment is

$$-\omega F dx. \qquad (d)$$

(The horizontal components, being of the order of $\left(\frac{dy}{dx}\right)^2$, are neglected.)

175. Now in the actual condition of equilibrium all these forces are balanced by those which we have supposed to act on the interior mass of the slice, namely, a vertical force $\rho\omega Y dx$, and a couple of which the moment is $\rho\omega L dx$. Hence we must have

$$(b) + (c) + \rho\omega Y dx = 0,$$

$$(a) + (d) + \rho\omega L dx = 0.$$

Introducing the actual values of (a), (b), (c), (d), and observing that, since we neglect $\left(\frac{dy}{dx}\right)^2$, we may put $\frac{1}{R} = -\frac{d^2y}{dx^2}$, we find, after dividing by ωdx,

$$T\frac{d^2y}{dx^2} - \frac{dF}{dx} + \rho Y = 0, \qquad (1)$$

$$(q+T)\,\kappa^2\frac{d^3y}{dx^3}-F+\rho L=0. \qquad (2)$$

In order to eliminate the unknown F, we have only to subtract, after differentiating (2). We thus obtain, finally,

$$(q+T)\,\kappa^2\frac{d^4y}{dx^4}-T\frac{d^2y}{dx^2}+\rho\frac{dL}{dx}-\rho Y=0 \qquad (3)$$

as the differential equation which must be satisfied in the condition of equilibrium.

176. We have no occasion to integrate the equation (3); but it is essential to ascertain the conditions which would serve to determine the arbitrary constants contained in its general solution. These are to be obtained from the data of the problem relative to the ends of the rod.

Now the forces acting on the surface of the left-hand face are only the given external forces (Art. **172**), and the interior tensions arising from extension, and these must balance one another. Hence we must have

$$F_{x=0}+F_0=0,$$

and (see end of Art. **173**)

$$-(q+T)\,\kappa^2\left(\frac{d^2y}{dx^2}\right)_{x=0}+G_0=0. \qquad (4)$$

The first of these, combined with (2), gives

$$(q+T)\,\kappa^2\left(\frac{d^3y}{dx^3}\right)_{x=0}+F_0+\rho\,(L)_{x=0}=0. \qquad (5)$$

These equations ((4) and (5)) will furnish the required conditions. (It is evident that similar equations must subsist at the other end of the rod.)

177. In order to form the differential equation of motion, we have now only to substitute in (3) the forces arising from the resistances of the particles to acceleration, instead of those supposed to act on the interior of the mass.

Hence, instead of the vertical force $\rho\,\omega\,Y\,dx$, supposed to act on the mass of a slice, we must substitute $-\rho\,\omega\,\dfrac{d^2y}{dt^2}\,dx$ or $-\dfrac{d^2y}{dx^2}$ instead of Y.

And instead of the supposed couple $\rho\,\omega\,L\,dx$, we must substitute one which is found as follows:

Since the particles in a plane transverse section remain in a plane section, and since the inclination of the plane of the section to the vertical is $\frac{dy}{dx}$, if η be the ordinate of a particle dm, reckoned (as in Art. **173**) in the plane of the section from its horizontal principal axis, the angular velocity of the plane being $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ (in the direction of positive rotation), the linear velocity of dm (estimated from right to left) is $\eta\,\frac{d}{dt}\frac{dy}{dx}$, and therefore its resistance to acceleration (in the same direction) is $-\eta\left(\frac{d}{dt}\right)^2\frac{dy}{dx}\,.\,dm$, and the moment of this resistance, with its proper sign, is $-\eta^2\left(\frac{d}{dt}\right)^2\frac{dy}{dx}\,.\,dm$. Now, considering dm as the mass of an element of an infinitesimal slice, we have $dm = \rho\,d\omega\,dx$; hence the above moment is

$$-\eta^2\,\frac{d^3y}{dt^2\,dx}\,\rho\,d\omega\,dx\,;$$

and taking the sum of all such moments for the whole slice, we have

$$-\rho\,\frac{d^3y}{dt^2\,dx}\,dx\int\eta^2\,d\omega = -\kappa^2\,\rho\,\omega\,\frac{d^3y}{dt^2dx}\,dx$$

as the expression to be substituted for $\rho\,\omega\,L\,dx$. Hence we must put $-\kappa^2\frac{d^3y}{dt^2\,dx}$ instead of L.

Making these substitutions in (3) we obtain

$$\kappa^2\left((q+T)\,\frac{d^4y}{dx^4}-\rho\,\frac{d^4y}{dt^2dx^2}\right)-T\,\frac{d^2y}{dx^2}+\rho\,\frac{d^2y}{dt^2}=0 \qquad (6)$$

as the differential equation required[1].

[1] See Klebsch, *Theorie der Elasticität fester Körper*, § 61, where this equation (with a different notation) is deduced, as a particular case, from the general theory of elastic solids. The equation usually given in elementary works does not contain the term $\frac{d^4y}{dt^2dx^2}$, which arises from the angular motion of the sections of the rod. (See, for example, Poisson, *Traité de Mécanique*, tom. ii. § 5.) It may in fact be neglected without sensible error in ordinary cases.

178. This may be put in a somewhat more convenient form as follows:

Put $q+T=b^2\rho$, $T=a^2\rho$, and it becomes

$$\kappa^2\left(b^2\frac{d^4y}{dx^4}-\frac{d^4y}{dt^2\,dx^2}\right)-a^2\frac{d^2y}{dx^2}+\frac{d^2y}{dt^2}=0. \qquad (6')$$

(In order to see the homogeneity of this equation it is desirable to observe the meanings of a and b. $T\omega$ is the actual tension, and ρ the actual density, in the axis of the rod. $q\omega$ is the tension which would have to be applied to the rod in its natural state in order to double its length, if the law of extension held good without limit. Hence $T\omega$ and $q\omega$ are forces, and can be represented by weights, say by the weights of lengths λ and λ' of the rod, taken at its actual density ρ. Then $T\omega=g\rho\lambda\omega$, $q\omega=g\rho\lambda'\omega$, and therefore

$$g(\lambda+\lambda')=b^2, \qquad g\lambda=a^2;$$

so that a^2, b^2 are the half squares of the velocities which would be acquired by a heavy body falling vertically down distances λ, $\lambda+\lambda'$. Hence a and b are of one dimension in space and -1 in time.)

179. In order to find particular integrals of (6′) we assume

$$y=u\cos\frac{m}{\kappa}t+v\sin\frac{m}{\kappa}t, \qquad (7)$$

u and v being functions of x to be determined.

Substituting this value of y in (6′), we find an equation of the form

$$U\cos\frac{m}{\kappa}t+V\sin\frac{m}{\kappa}t=0,$$

in which U and V do not contain t; so that in order to satisfy the equation we must have separately

$$U=0, \qquad V=0.$$

These equations are exactly similar in form, and we therefore need only consider one of them. The first is

$$\kappa^2 b^2\frac{d^4u}{dx^4}+(m^2-a^2)\frac{d^2u}{dx^2}-\frac{m^2}{\kappa^2}u=0; \qquad (8)$$

which, being linear with constant coefficients, can be integrated in the usual way. The general solution is

$$u=A\epsilon^{k_1x}+B\epsilon^{k_2x}+C\epsilon^{k_3x}+D\epsilon^{k_4x},$$

where k_1, k_2, k_3, k_4 are the roots of the equation

$$\kappa^2 b^2 k^4 + (m^2 - a^2)\, k^2 - \frac{m^2}{\kappa^2} = 0\,; \tag{9}$$

which gives $$k^2 = \frac{1}{2\,\kappa^2 b^2}\{a^2 - m^2 \pm ((a^2 - m^2)^2 + 4\, m^2 b^2)^{\frac{1}{2}}\}. \tag{9'}$$

We may represent the four values of k in the form

$$\pm\,\phi_1(m), \qquad \pm\,\phi_2(m)\,;$$

and, making for convenience a change in the meaning of the constants A, B, C, D, we may put the values of u and v in the form

$$u = A\,\frac{\epsilon^{\phi_1(m)\,x} + \epsilon^{-\phi_1(m)\,x}}{2} + B\,\frac{\epsilon^{\phi_1(m)\,x} - \epsilon^{-\phi_1(m)\,x}}{2}$$
$$+ C\,\frac{\epsilon^{\phi_2(m)\,x} + \epsilon^{-\phi_2(m)\,x}}{2} + D\,\frac{\epsilon^{\phi_2(m)\,x} - \epsilon^{-\phi_2(m)\,x}}{2}\,; \tag{10}$$

v = a similar expression with different constants, say A', B', C', D'.

180. Introducing these values of u and v in (7), we obtain a value of y, which is a particular integral of (6′), and in which all the constants, including m, are arbitrary, and would remain so if we had only to satisfy (6′). But in every actual problem we have also to satisfy the terminal conditions; and these conditions lead to an equation of which the roots are the only admissible values of m, besides other equations which *partly* determine the constants A, B, &c.

Before investigating these conditions however in particular cases, we will examine more closely the nature of the four values of k. Since the last term of (9) is negative, one value of k^2 is necessarily positive, and the other negative. Suppose we call the positive value α^2 and the negative value $-\beta^2$, then we shall have

$$k = \pm\,\alpha, \qquad k = \pm\,\beta\sqrt{-1}$$

for the four values of k; and thus we may put (10) in the form

$$u = A\,\frac{\epsilon^{\alpha x} + \epsilon^{-\alpha x}}{2} + B\,\frac{\epsilon^{\alpha x} - \epsilon^{-\alpha x}}{2} + C\cos\beta x + D\sin\beta x, \tag{11}$$

in which it is important to remember that α, β are determinate functions of m, given by (9), while the value of m itself has still to be determined.

181. We will take now the case which is in some respects the most simple, namely, that in which the ends of the axis of the rod are fixed, but the terminal faces are subject to no other constraint. (The tension T, which must be supposed to be applied to them on every unit of surface, will evidently leave the *directions* of the faces free.) This case is of practical interest, because it may be taken to represent that of a wire (e. g. a pianoforte string) of which the vibrating part is determined by stretching it over bridges.

Now referring to the terminal equation (4) (Art. **176**) we see that on the suppositions now made we must put $G_0 = 0$, and consequently

$$\frac{d^2y}{dx^2} = 0$$

at both ends of the rod. (The equation (5) gives no condition. It would merely serve to determine the value of the pressure $F_0\omega$ supported by the point to which the end of the axis is fixed.)

But the fixity of the ends of the axis gives us two more conditions, namely,

$$y = 0$$

at both ends. We have then to put in (7) the value (11) for u, and a similar value for v, with A', B', &c. instead of A, B, &c., and then express the conditions that

$$y = 0 \quad \text{and} \quad \frac{d^2y}{dx^2} = 0,$$

both when $x = 0$ and when $x = l$, for all values of t. These conditions, relatively to $x = 0$, give, as will be easily found,

$$A + C = 0, \qquad A' + C' = 0,$$
$$\alpha^2 A - \beta^2 C = 0, \qquad \alpha^2 A' - \beta^2 C' = 0;$$

which it is impossible to satisfy otherwise than by

$$A = 0, \quad C = 0, \quad A' = 0, \quad C' = 0.$$

The conditions relatively to $x = l$ thus become simplified, and are

$$B\frac{\epsilon^{\alpha l} - \epsilon^{-\alpha l}}{2} + D\sin\beta l = 0,$$

$$\alpha^2 B\frac{\epsilon^{\alpha l} - \epsilon^{-\alpha l}}{2} - \beta^2 D\sin\beta l = 0,$$

with similar equations for B' and D'. These give

$$B\frac{\epsilon^{al}-\epsilon^{-al}}{2}=0, \qquad D\sin\beta l=0;$$

now the factor multiplied by B cannot vanish, since a is real. Hence we must have

$$B=0, \quad B'=0, \quad \sin\beta l=0.$$

Hence D and D' remain arbitrary, while β must satisfy the equation $\sin\beta l=0$, which gives

$$\beta=\frac{i\pi}{l}.$$

182. The values of u, v (see equation (11)) are now reduced to $u=D\sin\beta x$, $v=D'\sin\beta x$, and therefore from (7)

$$y=\sin\beta x\left(D\cos\frac{mt}{\kappa}+D'\sin\frac{mt}{\kappa}\right),$$

in which β may have any of the infinite series of values $\frac{i\pi}{l}$ obtained by giving all values to the integer i from 1 to ∞, and to each value of β corresponds a value of m obtained from (9) by putting $k^2=-\beta^2$. Hence, taking the sum of all the particular values of y, we have for the general solution of (6′) appropriate to this problem

$$y=\sum_{i=1}^{i=\infty}\sin\frac{i\pi x}{l}\left(C_i\cos\frac{m_i t}{\kappa}+D_i\sin\frac{m_i t}{\kappa}\right); \qquad (12)$$

where C_i, D_i are arbitrary constants in the usual sense, that is, depend only on initial displacements and velocities. (It will be easily seen that it is useless to include negative values of i, since m_i (see next article) only depends on i^2.)

183. Solving equation (9) for m^2, we find, after slight reductions,

$$\frac{m^2}{\kappa^2}=-k^2\,\frac{a^2-\kappa^2 b^2 k^2}{1-\kappa^2 k^2};$$

and since, for real values of β,

$$k^2=-\beta^2=-\frac{i^2\pi^2}{l^2},$$

the above formula gives

$$\frac{m_i^2}{\kappa^2}=\frac{i^2\pi^2}{l^2}\cdot\frac{l^2a^2+i^2\pi^2\kappa^2b^2}{l^2+i^2\pi^2\kappa^2}. \qquad (13)$$

Now (12) shews that the vibration is compounded of simple harmonic vibrations of which the periods are the values of $\frac{2\pi\kappa}{m_i}$; or the number of vibrations, in a unit of time, of the i^{th} component tone, is $\frac{m_i}{2\pi\kappa}$. Calling this number n_i, we have from (13),

$$n_i = \frac{i}{2l}\left(\frac{l^2 a^2 + i^2\pi^2\kappa^2 b^2}{l^2 + i^2\pi^2\kappa^2}\right)^{\frac{1}{2}}. \tag{14}$$

This shews that in general the component tones do not belong to a harmonic scale [1].

184. Let us however examine some special cases.

If we suppose the rod infinitely thin we may neglect κ altogether. The differential equation (6′) then reduces itself to

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2},$$

the ordinary equation for a perfectly flexible string; and (14) gives $n_i = \frac{ia}{2l}$, the value found before.

But if we consider the rod as very thin, without being infinitely thin, so that $\frac{\kappa^2}{l^2}$ is a very small fraction, the value (14) will be applicable to the case of a metallic string or wire. In this case, neglecting the square of $\frac{\kappa^2}{l^2}$, and assuming the section of the wire to be a circle with radius r, so that $\kappa^2 = \frac{1}{4}r^2$, we find

$$n_i = \frac{ia}{2l}\left\{1 + \frac{i^2\pi^2}{8}\frac{r^2}{l^2}\left(\frac{b^2}{a^2} - 1\right)\right\};$$

hence if we put $\frac{ia}{2l} = N_i$ (the number of vibrations calculated on the supposition of infinite thinness or perfect flexibility [2]), and put for b^2 and a^2 their values (Art. **178**), we have

[1] The process which has been given in Arts. **179**–**183** is substantially the same as that of Klebsch, § 61.

[2] Strickly speaking, the supposition of infinite thinness ought to be distinguished from that of perfect flexibility. We can imagine a thick string of which only the central infinitely thin axis should resist extension or contraction. Such a string might be regarded as perfectly flexible. But the

$$n_i = N_i\left\{1 + \frac{i^2\pi^2}{8}\frac{r^2}{l^2}\cdot\frac{q}{T}\right\},$$

which gives what is called the *correction for rigidity.*

This correction may be put in another form thus: from (14) we have

$$n_i = \frac{i}{2l}\left(a^2 + i^2\pi^2\frac{\kappa^2}{l^2}(b^2 - a^2)\right)^{\frac{1}{2}} \text{ nearly,}$$

$$= \frac{i}{2l}\left\{\frac{1}{\rho}\left(T + i^2\frac{\pi^2 r^2 q}{4l^2}\right)\right\}^{\frac{1}{2}};$$

so that for a given value of i, that is, for a tone of given order, the number of vibrations corresponding to any actual tension T may be calculated as if the string were perfectly flexible, by substituting for T a fictitious tension

$$= T + i^2\frac{\pi^2 r^2 q}{4l^2}.$$

The term thus added is sensibly independent of T, since l (the actual length between the bridges) is constant, and r is sensibly invariable, at least for moderate variations of T. If the tension be supplied by a weight W, then $W = T\omega$. Suppose Q is the weight which would double the length of the string if the law of extension held good indefinitely, then $Q = q\omega$. Hence the fictitious weight to be substituted in calculation for the actual weight is

$$W + i^2\frac{\pi^2 r^2}{4l^2}Q.$$

It would be difficult to calculate the value of the added term *à priori*, because the values of the very small ratio $\left(\frac{r}{l}\right)^2$ and of the very large weight Q, could hardly be obtained with sufficient accuracy; but it is easily ascertained experimentally by com-

inertia of the outer parts would introduce the term $\frac{d^4y}{dt^2\,dx^2}$ in the differential equation, though the term $\frac{d^4y}{dx^4}$, which arises from the resistance of the outer parts to extension or contraction, would disappear. But the strings used for musical purposes never approximate to this character, though the converse arrangement is common, e.g. in guitar strings made by winding fine wire upon a silk core.

paring the tones produced by two different weights. The tones corresponding to other values of W can then be calculated, and they are found on trial to agree very exactly with those actually produced.

185. We will next suppose that, the ends of the axis being still fixed, the distance between them is the natural length of the rod, so that $T = 0$; hence $a = 0$, and (14) becomes

$$n_i = \tfrac{1}{2}\pi i^2 \frac{\kappa}{l} \cdot \frac{b}{(l^2 + i^2\pi^2\kappa^2)^{\frac{1}{2}}}$$

where $\qquad b^2 = \dfrac{q}{\rho}.$

In this case, since κ is small, the values of n_i are, for moderate values of i, sensibly proportional to the series of square numbers 1^2, 2^2, 3^2, &c., so that the component tones are the 1st, 4th, 9th, &c. of a harmonic scale. Thus the first of the upper tones is two octaves above the fundamental tone; and the second is one octave and a major second above the first; &c.

The supposition now made may be approximately realized in several ways, of which the simplest consists in merely laying a rod (e. g. a bar of steel) upon two bridges placed as close as possible to its ends.

186. If, instead of merely supposing the ends of the axis fixed, we had supposed the planes of the terminal faces of the rod to be fixed, then, instead of the simple formula (13) which gives the values of m, we should have found a very complicated transcendental equation. The same thing happens if one terminal face be fixed and the other entirely free, or if both be entirely free. In the two latter cases this equation is always somewhat simplified by the circumstance that $T = 0$, and therefore $a = 0$. But it becomes much more simplified if we neglect the term $\dfrac{d^4y}{dt^2\,dx^2}$ in the differential equation, which we may generally do without sensible error. We shall therefore introduce this simplification in what follows.

187. Since the differential equation was founded on the hypothesis that the particles which in the undisturbed state are in a plane at right angles to the axis continue to be so at all times, if a terminal face of the rod be fixed, the axis at that end must always be at right angles to it; and as the end of the axis itself is fixed, we must have, at that end,

$$y = 0, \qquad \frac{dy}{dx} = 0.$$

These therefore are the terminal conditions for a fixed face.

But if a terminal face be entirely free, we must obtain the terminal conditions from equations (4) and (5) (Art. **176**). Now, at a free end, G_0 (or G_1) and F_0 (or F_1) are both 0. Also L, in (5), arises (see Art. **177**) from the angular motion of the planes of the elementary slices, the effect of which we are now going to neglect; hence these equations give

$$\frac{d^2y}{dx^2} = 0, \quad \frac{d^3y}{dx^3} = 0$$

as the terminal conditions at a free end.

We have already seen that the conditions are

$$y = 0, \qquad \frac{d^2y}{dx^2} = 0$$

at an end where only the extremity of the axis is fixed, so that the *direction* of the plane of the face is free.

There are altogether six possible combinations, of which we have already considered one. Of the remaining five we shall only examine the three which are of most importance, namely, both faces fixed, both free, one fixed and the other free.

188. The equation (6′), if we omit the second term, and put $a = 0$ (since we suppose $T = 0$), becomes

$$\frac{d^2y}{dt^2} + \kappa^2 b^2 \frac{d^4y}{dx^4} = 0. \tag{15}$$

To find particular solutions of this equation, we may conveniently assume

$$y = u \cos \frac{\kappa b}{l^2} m^2 t + v \sin \frac{\kappa b}{l^2} m^2 t; \tag{16}$$

in which u, v are to be functions of x, l is the length of the rod, and m a constant to be determined. Substituting this value of y in (15), we find that in order to satisfy that equation for all values of t, we must have

$$\frac{d^4u}{dx^4} = \frac{m^4}{l^4} u, \qquad \frac{d^4v}{dx^4} = \frac{m^4}{l^4} v.$$

The general solution of the first of these equations may evidently be written in the form

$$u = A \cos \frac{mx}{l} + B \sin \frac{mx}{l}$$

$$+ C \frac{\epsilon^{\frac{mx}{l}} + \epsilon^{-\frac{mx}{l}}}{2} + D \frac{\epsilon^{\frac{mx}{l}} - \epsilon^{-\frac{mx}{l}}}{2}; \qquad (17)$$

and v will be given by a similar equation, with other constants A', B', C', D'.

It will save much trouble to adopt the following abbreviated notation.

Let $$\frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2} = \sigma(\theta), \qquad \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2} = \delta(\theta).$$

Then we shall evidently have,

$$\sigma(\theta) = \sigma(-\theta), \qquad \delta(\theta) = -\delta(-\theta), \qquad \sigma(0) = 1,$$

$$\delta(0) = 0, \qquad \frac{d}{d\theta}\sigma(n\theta) = n\,\delta(n\theta), \qquad \frac{d}{d\theta}\delta(n\theta) = n\,\sigma(n\theta).$$

189. Thus the equation (17) becomes

$$u = A \cos \frac{mx}{l} + B \sin \frac{mx}{l} + C\sigma\left(\frac{mx}{l}\right) + D\delta\left(\frac{mx}{l}\right); \quad (18)$$

and we have now to find the values of the constants which will satisfy the terminal conditions in each case.

First, then, let us suppose both ends entirely free. The conditions (see Art. **187**) are

$$\frac{d^2y}{dx^2} = 0, \qquad \frac{d^3y}{dx^3} = 0;$$

both when $x = 0$ and when $x = l$; and since these are to be satisfied for all values of t, it is evident that we must have

$$\frac{d^2u}{dx^2} = 0, \qquad \frac{d^2v}{dx^2} = 0, \qquad \frac{d^3u}{dx^3} = 0, \qquad \frac{d^3v}{dx^3} = 0.$$

Putting then $x = 0$ and $x = l$ successively in the values of these differential coefficients deduced from (18) and from the corresponding expression for v, we find (for $x = 0$)

$$-A + C = 0, \qquad -B + D = 0;$$

so that

$$u = A\left(\cos \frac{mx}{l} + \sigma\left(\frac{mx}{l}\right)\right) + B\left(\sin \frac{mx}{l} + \delta\left(\frac{mx}{l}\right)\right);$$

and then the conditions relative to $x = l$ become

$$\left.\begin{aligned} A\,(-\cos m + \sigma(m)) + B\,(-\sin m + \delta(m)) &= 0, \\ A\,(\sin m + \delta(m)) + B\,(-\cos m + \sigma(m)) &= 0; \end{aligned}\right\} \qquad (19)$$

from which, eliminating A and B, we have

$$(\sigma(m) - \cos m)^2 = (\delta(m))^2 - \sin^2 m;$$

now, by the definition of $\sigma(m)$ and $(\delta(m))$,

$$(\sigma(m))^2 - (\delta(m))^2 = 1,$$

hence this equation becomes

$$\sigma(m)\cos m = 1, \quad \text{or}$$

$$\frac{\epsilon^m + \epsilon^{-m}}{2}\cos m = 1. \tag{20}$$

The roots of this equation are the admissible values of m; and if we denote them by m_1, m_2, &c., and call A_i, B_i the corresponding values of A, B, either of the equations (19) gives the ratio $A_i : B_i$. We may therefore take

$$A_i = C_i(\sin m_i - \delta(m_i)),$$
$$B_i = -C_i(\cos m_i - \sigma(m_i)),$$

where C_i is arbitrary. Thus we shall have from (18)

$$u_i = C_i X_i, \quad \text{where}$$

$$X_i = (\sin m_i - \delta(m_i))\left(\cos\frac{m_i x}{l} + \sigma\left(\frac{m_i x}{l}\right)\right)$$
$$-(\cos m_i - \sigma(m_i))\left(\sin\frac{m_i x}{l} + \delta\left(\frac{m_i x}{l}\right)\right); \tag{21}$$

and $v_i = D_i X_i$, where D_i is another arbitrary constant. The general value of y, which is the sum of all particular values, will then be

$$y = \Sigma X_i\left(C_i\cos\frac{\kappa b}{l^2}m_i^2 t + D_i\sin\frac{\kappa b}{l^2}m_i^2 t\right); \tag{22}$$

and this is the equation expressing the vibration of a rod free at both ends.

The constants C_i, D_i are determined by the initial displacements and velocities, in a manner which will be explained afterwards.

190. If instead of supposing the ends of the rod entirely free, we suppose both the terminal faces entirely fixed, the terminal conditions are (Art. **187**)

$$y = 0, \qquad \frac{dy}{dx} = 0,$$

both when $x = 0$ and when $x = l$.

Assuming then (16) and (18) as before, and proceeding exactly as in the last Article, we find the same equation (20) for the determination of the values of m_i, but

$$A_i = \quad C_i\,(\sin m_i - \delta\,(m_i)),$$
$$B_i = -C_i\,(\cos m_i - \sigma\,(m_i)),$$

and
$$y = \Sigma\, Y_i\left(C_i \cos\frac{\kappa b}{l^2}\, m_i^2 t + D_i \sin\frac{\kappa b}{l^2}\, m_i^2 t\right), \tag{23}$$

where
$$Y_i = (\sin m_i - \delta\,(m_i))\left(\cos\frac{m_i x}{l} - \sigma\left(\frac{m_i x}{l}\right)\right)$$
$$-(\cos m_i - \sigma\,(m_i))\left(\sin\frac{m_i x}{l} - \delta\left(\frac{m_i x}{l}\right)\right). \tag{24}$$

Comparing the expressions (22), (23), we see that the component tones have the same pitch, whether the terminal faces be both free or both fixed. For the values of m_i are the roots of the same equation (20) in both cases, and the number of vibrations in a unit of time, for the tone of the i^{th} order, is

$$n_i = \frac{m_i^2}{2\,\pi} \cdot \frac{\kappa b}{l^2}\,.$$

The constant b depends (Art. **178**) only on the material of which the rod is made, and m_i is an abstract number, independent both of material and dimensions. Hence, when the material is given, the number of vibrations, for a tone of given order, varies inversely as the square of the length of the rod, and directly as the radius of gyration of the sectional area about that diameter which is at right angles to the plane of vibration.

If the section is elliptic or rectangular, then κ is simply proportional to the thickness measured in the plane of vibration.

191. In the last Article it was supposed that both the terminal faces were fixed, but that there was no permanent tension, so that the natural length of the axis was maintained.

The supposition of permanent tension, with fixed terminal faces, leads to much more complicated equations, but they may be treated in an approximate manner in the only case of practical importance, namely, that in which the thickness of the rod is very small compared with its length. The result may then be considered as giving the *correction for rigidity* for a wire, or for a long and thin lamina, not stretched over bridges, but firmly clamped at the ends.

We may take in this case the equations (7), (9), and (11), as in Arts. **179** and **180**. But we shall suppose the term $\dfrac{d^4 y}{dt^2\,dx^2}$ in (6′) to be neglected, so that instead of (9′) we get the simpler form
$$2\,\kappa^2\, b^2\, k^2 = a^2 \pm (a^4 + 4\, m^2\, b^2)^{\frac{1}{2}}\,;$$

and therefore, since a^2 and $-\beta^2$ are the two values of k^2, we may write the value of a^2 thus:

$$a^2 = \frac{\left(1 + 4\frac{m^2 l^2}{\kappa^2 a^2} \cdot \frac{\kappa^2}{l^2} \cdot \frac{b^2}{a^2}\right)^{\frac{1}{2}} + 1}{2\kappa^2 \frac{b^2}{a^2}};$$

and β^2 will be given by changing $+1$ into -1 in the last term of the numerator.

Now in the case of a metallic wire or lamina, $\frac{b^2}{a^2}$ is a large number (see Art. **178**) since T is small compared with q. But $\frac{\kappa^2}{l^2}$ is very small; and the legitimacy of the following approximation depends upon the assumption that $\frac{\kappa^2}{l^2}$ is so small that $\frac{\kappa^2}{l^2} \cdot \frac{b^2}{a^2}$ is also very small. From this assumption it follows that $a\,l$ is very large, since $a^2 l^2$ is expressed by a fraction in which the numerator is > 2 and the denominator is the small fraction $2\frac{\kappa^2}{l^2}\frac{b^2}{a^2}$.

Now the terminal conditions are $y = 0$, $\frac{dy}{dx} = 0$, at both ends; and from these, proceeding as in Art. **189**, we find from equation (11) $\quad A + C = 0, \qquad aB + \beta D = 0,$

$$A\,\sigma(a\,l) + B\,\delta(a\,l) + C\cos\beta l + D\sin\beta l = 0,$$

$$a\,(A\,\delta(a\,l) + B\,\sigma(a\,l)) + \beta\,(D\cos\beta l - C\sin\beta l) = 0;$$

and hence, eliminating A, B, C, D, and reducing by means of the identity $(\sigma(a\,l))^2 - (\delta(a\,l))^2 = 1$, we find, finally,

$$\frac{\delta(a\,l)\sin\beta l}{1 - \sigma(a\,l)\cos\beta l} + \frac{2a\beta}{a^2 - \beta^2} = 0; \qquad (m)$$

and if in this equation the values of a and β given above were introduced, we should obtain an equation in m, of which the roots would be the values of m_1, m_2, &c.

Now the values of a^2, β^2 give, as will be found at once without difficulty, $\frac{a\beta}{a^2 - \beta^2} = \frac{m\,b}{a^2}$. Also, since $a\,l$ is very large, we have, neglecting ϵ^{-al}, $\sigma(a\,l) = \delta(a\,l) = \frac{1}{2}\epsilon^{+al}$, and the equation (m) becomes

$$\frac{\frac{1}{2}\epsilon^{al}\sin\beta l}{1 - \frac{1}{2}\epsilon^{al}\cos\beta l} + \frac{2\,m\,b}{a^2} = 0;$$

or, $\epsilon^{-\alpha l}$ being again neglected,

$$\tan \beta l = \frac{2 m b}{a^2}.$$

Now the value of β^2 gives

$$l^2 \beta^2 = \frac{m^2 l^2}{\kappa^2 a^2} \text{ nearly}$$

(by developing the binomial as far as the second term); hence

$$l \beta = \frac{m l}{\kappa a} \quad \text{nearly.}$$

But the number of vibrations in a unit of time is $\frac{m}{2 \pi \kappa}$; and since the case differs very little from that of an infinitely thin string, this number differs very little from $\frac{i a}{2 l}$, so that $\frac{m l}{\kappa a}$ differs very little from $i \pi$, or $\beta l = i \pi + \theta$, where θ is very small; hence $\frac{2 m b}{a^2} = \tan \beta l = \tan \theta$ is very small; and we may take $\frac{2 m b}{a^2} = \theta$, and therefore

$$\beta l = i \pi + \frac{2 m b}{a^2};$$

and, equating this to $\frac{m l}{\kappa a}$, we have

$$m \left(\frac{l}{\kappa a} - \frac{2 b}{a^2}\right) = i \pi;$$

or, introducing the subscript index to distinguish the different values of m,

$$\frac{m_i}{\kappa} = \frac{i \pi a}{l} \left(1 - 2 \frac{\kappa}{l} \frac{b}{a}\right)^{-1} = \frac{i \pi a}{l} \left(1 + 2 \frac{\kappa}{l} \frac{b}{a}\right) \quad \text{nearly.}$$

Let n_i be the number of vibrations, in a unit of time, of the i^{th} tone, and N_i the number calculated on the supposition of infinite thinness; then

$$N_i = \frac{i a}{2 l}, \quad \text{and} \quad n_i = \frac{m_i}{2 \pi \kappa}; \quad \text{hence}$$

$$n_i = N_i \left(1 + 2 \frac{\kappa}{l} \frac{b}{a}\right) = N_i \left(1 + 2 \frac{\kappa}{l} \left(1 + \frac{q}{T}\right)^{\frac{1}{2}}\right). \tag{n}$$

Comparing this with the corresponding expression deduced in Art. **184** on the supposition that the *directions* of the terminal faces were free, viz.

$$n_i = N_i \left(1 + \frac{i^2 \pi^2}{2} \frac{\kappa^2}{l^2} \cdot \frac{q}{T}\right), \tag{n'}$$

we see that they differ essentially, especially in this respect, that in the case (n) of *fixed faces* the pitch of all the component tones is raised, by the rigidity, through the same interval, so that they do not cease to form a harmonic series; whereas in the other case (n') each tone is raised through a greater interval than the next lower one, and the series is therefore no longer strictly harmonic.

An expression equivalent to (n), and obtained by nearly the same process, was given by Seebeck[1], and found by him to agree with experiment when the ends of the wire were clamped.

In the case of a wire stretched over bridges, the form (n') has been found to agree with experiment, in the manner mentioned at the end of Art. **184.** But the deviation of the upper tones from the harmonic scale is probably too small to be made sensible to the ear.

192. The last of the cases which we proposed to examine is that in which one terminal face (suppose that at which $x = 0$) is fixed and the other free. The conditions then are (Art. **187**)

$$y = 0, \qquad \frac{dy}{dx} = 0, \quad \text{when} \quad x = 0,$$

$$\frac{d^2y}{dx^2} = 0, \qquad \frac{d^3y}{dx^3} = 0, \quad \text{when} \quad x = l.$$

Again then, assuming (16) and (18), we find in the first place $A + C = 0$, $B + D = 0$, and then

$$A\,(\cos m + \sigma\,(m)) + B\,(\sin m + \delta\,(m)) = 0,$$

$$A\,(\sin m - \delta\,(m)) - B\,(\cos m + \sigma\,(m)) = 0;$$

eliminating A and B we obtain, after reduction,

$$\sigma\,(m) \cos m = -1, \quad \text{or}$$

$$\frac{\epsilon^m + \epsilon^{-m}}{2} \cos m = -1 \tag{25}$$

as the equation for determining the values of m; and we may take $A_i = C_i\,(\sin m_i + \delta\,(m_i))$, $\qquad B_i = -C_i\,(\cos m_i + \sigma\,(m_i))$;

so that the value of y will be

$$y = \Sigma Z_i \left(C_i \cos \frac{\kappa b}{l^2}\, m_i^2 t + D_i \sin \frac{\kappa b}{l^2}\, m_i^2\, t\right); \tag{26}$$

in which C_i, D_i are arbitrary, and

[1] See the memoir referred to below (Art. **205**).

$$Z_i = (\sin m_i + \delta(m_i)) \left(\cos \frac{m_i x}{l} - \sigma\left(\frac{m_i x}{l}\right)\right.$$

$$\left. - (\cos m_i + \sigma(m_i)) \left(\sin \frac{m_i x}{l} - \delta\left(\frac{m_i x}{l}\right)\right)\right). \qquad (27)$$

Since the equation (25) is not the same as (20), the periods of the component tones will not be the same as in the two former cases. But the law of their variation with the length and sectional area of a rod of given material is still the same as that stated at the end of Art. **190.**

193. To complete the solution of the problems considered in Arts. **187–192**, we should have first to find the roots of the equations (20) and (25), which determine the periods of the component tones, and then to find the values of x which satisfy the equations $X_i = 0$, $Y_i = 0$, for each root of (20), and $Z_i = 0$ for each root of (25), in order to ascertain the positions of the nodes corresponding to each tone. The required calculations, for small values of i, which belong to the most important tones, are troublesome; especially those which relate to the nodes. And we shall only give a sufficient specimen of them to enable the reader, who may be so disposed, to verify the results which will be given below.

First, then, we have to find the values of m_i, which are the roots of the two equations (see (20) and (25)),

$$\frac{\epsilon^x + \epsilon^{-x}}{2} \cos x = \pm 1; \qquad (28)$$

where the upper sign corresponds to the case of both ends fixed or both free, and the lower to that of one fixed and the other free.

It is evident on inspection that if m be any root of either equation (28), then $-m$, and $\pm m\sqrt{-1}$ are also roots; now observing that

$$\cos(-m\theta) = \cos m\theta, \qquad \sigma(-m\theta) = \sigma(m\theta),$$

$$\cos(\pm m\theta\sqrt{-1}) = \sigma(m\theta), \qquad \sigma(\pm m\theta\sqrt{-1}) = \cos m\theta,$$

$$\sin(-m\theta) = -\sin m\theta, \qquad \delta(-m\theta) = -\delta(m\theta),$$

$$\sin(\pm m\theta\sqrt{-1}) = \pm\sqrt{-1}\,.\,\delta(m\theta),$$

$$\delta(\pm m\theta\sqrt{-1}) = \pm\sqrt{-1}\,.\,\sin m\theta,$$

we see, on examining the forms of the functions X_i, Y_i, Z_i, that the effect of changing any one of the four values $\pm m_i$, $\pm m_i\sqrt{-1}$ into any other, will in every case be merely to multiply the function by one of the factors ± 1, $\pm \sqrt{-1}$; and consequently all the four terms in the value of y given by the four roots, can be united into one term of the form (22), (23), or (26), according to the case in question. It is therefore only necessary to consider the positive real roots of (28).

194. The position of the roots of (28) may be most clearly exhibited by a graphic construction. If we draw the curve of which the equation is

$$y = \frac{\epsilon^x + \epsilon^{-x}}{2} \cos x,$$

it will cut the positive axis of x at distances

$$\frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \quad \&c.,$$

from the origin, and the distances from the axis of y at which it cuts the two lines $y = \pm 1$, will be the positive roots of (28). The curve itself will consist of a series of unsymmetrical waves, of which the amplitudes increase without limit. In Fig. 2, the

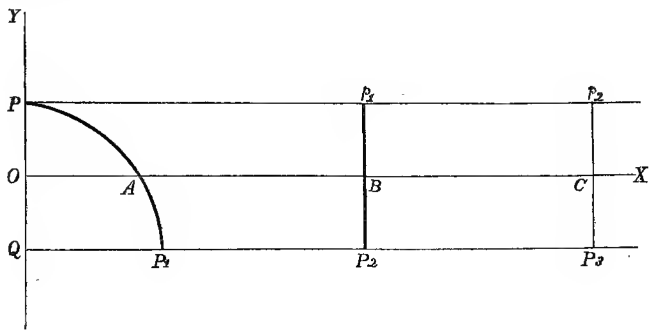

Fig. 2.

lines PP_1, p_1P_2, p_2P_3 represent portions of the curve included between the lines $y = \pm 1$, so that Pp_1, Pp_2 are two roots corresponding to the upper sign in (28), and QP_1, QP_2, QP_3 are three roots corresponding to the lower sign.

Since $\dfrac{\epsilon^x + \epsilon^{-x}}{2}$ increases indefinitely with x, it is evident that (28) requires $\pm \cos x$ to diminish indefinitely with x, so that for large values of i the values of m_i must approximate without limit to $\pm (2i + 1)\dfrac{\pi}{2}$. At the points A, B, C, &c., where

$$x = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \ \&c.,$$

the values of $\frac{dy}{dx}$ are alternately negative and positive, increasing numerically without limit. Hence the roots corresponding to the upper sign are alternately greater and less than $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, &c., and those corresponding to the lower sign alternately greater and less than $\frac{\pi}{2}$, $\frac{3\pi}{2}$, &c. On the scale to which the figure is drawn, the portion of curve $p_2 P_3$ is quite undistinguishable from the ordinate at C.

We will now shew how to calculate the values of QP_1, QP_2, Pp_1.

195. Suppose that in either of the equations

$$\frac{\epsilon^{x} + \epsilon^{-x}}{2} \cos x = \pm 1$$

we have found an approximate value of x, say $x = \mathrm{m}$. Then, assuming $\mathrm{m} + a$ as the true value, we have

$$\frac{\epsilon^{\mathrm{m}+a} + \epsilon^{-\mathrm{m}-a}}{2} \cos(\mathrm{m} + a) = \pm 1.$$

Developing this, and neglecting powers of a above the first, we find (using the notation explained in Art. **188**)

$$a = \frac{\cos \mathrm{m} \,.\, \sigma(\mathrm{m}) \mp 1}{\sin \mathrm{m} \,.\, \sigma(m) - \delta(m) \cos \mathrm{m}}. \tag{29}$$

If the value $\mathrm{m} + a$ thus found is not sufficiently exact, then it must be assumed as an approximation, and the process repeated, and so on as often as may be necessary.

We will take as an example the case of the fundamental tone of the rod with one end fixed. We then have to find the least positive root of (28), taking the lower sign; and we have seen (Art. **194**) that this is somewhat greater than $\frac{\pi}{2}$. Assuming then $x = \frac{\pi}{2} + a$ as a first approximation, and putting $\mathrm{m} = \frac{\pi}{2}$ in (29) we have

$$a = \frac{1}{\sigma\left(\frac{\pi}{2}\right)} = 0.398 \text{ nearly.}$$

Assuming therefore

$$x = \frac{\pi}{2} + 0.398 + a',$$

we shall find the value of a' by putting

$$m = \frac{\pi}{2} + 0.398$$

in (29) (taking the lower sign in the numerator). This gives

$$a' = -0.089 \quad \text{and} \quad x = 1.88 \text{ nearly.}$$

The next approximation gives

$$x = 1.8751,$$

which is sufficiently accurate.

196. For the higher tones of the rod with one end fixed, and for all the tones of the rod with both ends fixed or both free, the approximation is more rapid. The following are the results in the two cases

	I. One end fixed and one free.	II. Both ends fixed or both free.	
m_1	1.8751	4.7300	1
m_2	4.6940	7.8532	2,78
m_3	7.8548	10.9957	5,46
m_4	10.9955	14.1372.	9 –
			13.3 +

For still higher tones the formula

$$m_i = (2i \mp 1)\frac{\pi}{2}$$

may be used without sensible error, the upper sign belonging to case I. and the lower to case II.

The numbers of vibrations, being proportional to the values of m^2, are, for the higher tones, sensibly proportional to the squares of the odd numbers.

197. To find the interval between any two tones we may proceed as in the following example. The interval between the fundamental tone and the first upper tone, of the rod with one end fixed, is defined by the ratio $\left(\frac{4.6940}{1.8751}\right)^2$, of which the logarithm is 0.79704; and observing that $\log 6 = 0.77815$, we find that this ratio is equal to 6×1.0445 nearly. Now

$$1.0445 = 1 + \frac{1}{22+}\,\frac{1}{2+}\ \text{\&c.},$$

of which the first three convergents are $\frac{1}{1}, \frac{23}{22}, \frac{47}{45}$; hence 1.0445 exceeds $\frac{23}{22}$ by a fraction less than $\frac{1}{990}$, and the ratio of the interval is therefore a very little less than

$$6 \times \frac{23}{22} = \frac{4}{1} \times \frac{3}{2} \times \frac{23}{22}.$$

It follows that the interval is = two octaves + fifth + $(\frac{23}{22})$ nearly; and the interval $(\frac{23}{22})$ is a little less than $\frac{3}{4}$ths of a diatonic semitone, for $(\frac{16}{15})^{\frac{3}{4}} = \frac{21}{20}$ nearly.

Hence, if the fundamental tone were *C*, the first upper tone would be flatter than ♭ *a′* by a little more than a quarter of a diatonic semitone[1].

In this way we find the following to be the four first tones of a rod in the two cases, supposing the fundamental tone in each case to be *C*.

I.	II.
One end fixed and one free.	Both ends free or both fixed.
C	C
♭ *a′* −	♯*f* −
♭ *d‴* +	*f′* +
♭ $d^{(4)}$ +	*d″* −

The sign + signifies that the actual sound is somewhat sharper, and the sign − that it is somewhat flatter, than the tone indicated by the letter.

We find also that the ratio of the interval between the fundamental tone of a rod with one end fixed, and of the same rod with both ends free or fixed, is 6×1.106 nearly; so that the interval is a very little less than two octaves + fifth + minor second. Thus, if the fundamental tone were *C* in the first case, it would be a little flatter than *a′* in the second.

198. We shall now shew how to determine the position of the nodes in the several cases, and we will take first that of the rod with one end fixed.

Referring to equations (26), (27), (Art. **192**), we see that since at a node $y = 0$ for all values of t, the values of x, or the distances of the nodes from the fixed end, must be in this case roots of the equation $Z_i = 0$, namely, those positive roots which are less than l.

[1] A more systematic way of defining a small interval is to assign its ratio to the semitone of the 'equal temperament,' which is the twelfth part of an octave, and which we may call the 'mean semitone.' An interval of which the ratio is r contains $\dfrac{\log r}{\frac{1}{12}\log 2}$ mean semitones. Thus the interval (1.0445) contains $\dfrac{\log 1.0445}{\frac{1}{12}\log 2} = 0.787$ such semitones nearly. (See Art. **23**.)

The octave contains 10.74 diatonic semitones nearly; and the mean semitone is about 0.89 diatonic semitones. (Compare the table of intervals on p. 27.)

This equation (subscript indices being omitted) is

$$(\sin m + \delta\,(m))\left(\cos\frac{m\,x}{l} - \sigma\left(\frac{m\,x}{l}\right)\right)$$

$$-(\cos m + \sigma\,(m))\left(\sin\frac{m\,x}{l} - \delta\left(\frac{m\,x}{l}\right)\right) = 0, \qquad (30)$$

in which m is a determinate root (m_i) of the equation

$$\sigma\,(m)\cos m = -1. \qquad (31)$$

The equation (30) may be transformed as follows:
From (31) we have $\sigma\,(m_i) = -\sec m_i$, and therefore

$$(\delta\,(m_i))^2 = (\sigma\,(m_i))^2 - 1 = \tan^2 m_i.$$

Now we have seen (Art. **194**) that m_i is greater or less than $(2\,i - 1)\,\frac{\pi}{2}$ according as i is odd or even, so that we may put ($i = 0$ being excluded)

$$m_i = (2\,i - 1)\,\frac{\pi}{2} - (-)^i\,a_i \qquad (32)$$

where a_i is a small positive quantity, which diminishes indefinitely for increasing values of i.

Hence $\cos m_i$ is always negative, and $\sin m_i$ has the same sign as $\sin\,(2\,i - 1)\,\frac{\pi}{2}$, that is, $(-)^{i+1}$. Consequently, since $\delta\,(m_i)$ is necessarily positive, we must have

$$\delta\,(m_i) = (-)^i\tan m_i = \cos i\,\pi\tan m_i;$$

and therefore

$$\frac{\sin m_i + \delta\,(m_i)}{\cos m_i + \sigma\,(m_i)} = \frac{\sin m_i + \cos i\,\pi\tan m_i}{\cos m_i - \sec m_i} = -\frac{\cos m_i + \cos i\,\pi}{\sin m_i + \sin i\,\pi}$$

$$= -\cot\frac{m_i + i\,\pi}{2};$$

and (30) is therefore easily seen to become (indices being omitted)

$$\cos\left(\frac{m\,x}{l} - \frac{m + i\,\pi}{2}\right) - \sigma\left(\frac{m\,x}{l}\right)\cos\frac{m + i\,\pi}{2}$$

$$-\delta\left(\frac{m\,x}{l}\right)\sin\frac{m + i\,\pi}{2} = 0;$$

which is reducible, by means of the identities

$$\cos\theta + \sin\theta = \sqrt{2}\,.\sin\left(\theta + \frac{\pi}{4}\right),\quad \cos\theta - \sin\theta = \sqrt{2}\,.\cos\left(\theta + \frac{\pi}{4}\right),$$

to the form

$$\sqrt{2} \, . \cos\left(\frac{mx}{l} - \frac{m+i\pi}{2}\right) - \epsilon^{\frac{mx}{l}} \sin\left(\frac{m+i\pi}{2}+\frac{\pi}{4}\right)$$

$$-\epsilon^{-\frac{mx}{l}} \cos\left(\frac{m+i\pi}{2}+\frac{\pi}{4}\right) = 0. \quad (30')$$

Now let the origin of abscissæ be removed to the middle of the axis of the rod, by writing $x + \frac{l}{2}$ instead of x, so that x will now mean the distance of a node from the middle point. The above equation then becomes

$$\sqrt{2} \, . \cos\left(\frac{mx}{l} - \frac{i\pi}{2}\right) - \epsilon^{\frac{mx}{l}} \, . \, \epsilon^{\frac{m}{2}} \sin\left(\frac{m+i\pi}{2}+\frac{\pi}{4}\right)$$

$$-\epsilon^{-\frac{mx}{l}} \, . \, \epsilon^{-\frac{m}{2}} \cos\left(\frac{m+i\pi}{2}+\frac{\pi}{4}\right) = 0. \quad (33)$$

Now from equation (31), which is

$$\frac{\epsilon^m + \epsilon^{-m}}{2} = -\sec m,$$

we get (by adding and subtracting 1)

$$\tfrac{1}{4}\left(\epsilon^{\frac{m}{2}} + \epsilon^{-\frac{m}{2}}\right)^2 = -\sec m \sin^2 \frac{m}{2},$$

$$\tfrac{1}{4}\left(\epsilon^{\frac{m}{2}} - \epsilon^{-\frac{m}{2}}\right)^2 = -\sec m \cos^2 \frac{m}{2};$$

and since $\sec m$ is always negative, whilst the sine and cosine of $\frac{m}{2}$ have the same signs as the sine and cosine of $(2i-1)\frac{\pi}{4}$ (as is evident from (32)), these equations give

$$\epsilon^{\frac{m}{2}} + \epsilon^{-\frac{m}{2}} = 2\sqrt{-\sec m} \, . \sin\frac{m}{2} \sin(2i-1)\frac{\pi}{4},$$

$$\epsilon^{\frac{m}{2}} - \epsilon^{-\frac{m}{2}} = 2\sqrt{-\sec m} \, . \cos\frac{m}{2} \cos(2i-1)\frac{\pi}{4};$$

and thence, by addition and subtraction,

$$\epsilon^{\frac{m}{2}} = \sqrt{-\sec m} \, . \cos\left(\frac{m}{2} - \frac{i\pi}{2} + \frac{\pi}{4}\right)$$

$$-\epsilon^{-\frac{m}{2}} = \sqrt{-\sec m} \, . \cos\left(\frac{m}{2} + \frac{i\pi}{2} - \frac{\pi}{4}\right).$$

Introducing these values in (33), and observing the identities

$$\sin\theta\cos\phi = \tfrac{1}{2}(\sin(\theta+\phi)+\sin(\theta-\phi)),$$
$$\cos\theta\cos\phi = \tfrac{1}{2}(\cos(\theta+\phi)+\cos(\theta-\phi)),$$

we find, after obvious reductions,

$$\sqrt{2}\,.\cos\left(\frac{mx}{l}-\frac{i\pi}{2}\right)+\tfrac{1}{2}\sqrt{-\cos m}\,.\left(\epsilon^{\frac{mx}{l}}-\epsilon^{-\frac{mx}{l}}\cos i\pi\right)=0;$$

but from (32) it is evident that

$$\cos m_i = -\sin a_i;$$

hence the equation becomes

$$\cos\left(\frac{m_i x}{l}-\frac{i\pi}{2}\right)+\tfrac{1}{2}\sqrt{\tfrac{1}{2}\sin a_i}\,.\left(\epsilon^{\frac{m_i x}{l}}-\epsilon^{-\frac{m_i x}{l}}\cos i\pi\right)=0. \quad (34)$$

Another form is obtained from (33) by substituting for m_i, under the sine and cosine in the last two terms, the value (32). This will easily be found to give

$$\sqrt{2}\,.\cos\left(\frac{m_i x}{l}-\frac{i\pi}{2}\right)+\epsilon^{m_i\left(\frac{x}{l}+\frac{1}{2}\right)}\sin\tfrac{1}{2}a_i$$
$$-\epsilon^{-m_i\left(\frac{x}{l}+\frac{1}{2}\right)}\cos i\pi\cos\tfrac{1}{2}a_i=0. \quad (34')$$

199. To find the places of the nodes when both ends are free, we have (see Art. **189**) to solve the equation

$$(\sin m-\delta(m))\left(\cos\frac{mx}{l}+\sigma\left(\frac{mx}{l}\right)\right)$$
$$-(\cos m-\sigma(m))\left(\sin\frac{mx}{l}+\delta\left(\frac{mx}{l}\right)\right)=0; \quad (35)$$

in which m is a determinate root (m_i) of the equation

$$\sigma(m)=\sec m;$$

and (Art. **194**), $i=0$ being excluded,

$$m_i=(2i+1)\frac{\pi}{2}-(-)^i\beta_i, \quad (36)$$

where β_i is a small positive quantity, which diminishes indefinitely for increasing values of i.

In this case $\cos m_i$ is always positive, and $=\sin\beta_i$; and, proceeding as in the last Article, we find

$$\delta(m_i)=\tan m_i\cos i\pi; \quad \text{also}$$

$$\frac{\sin m_i - \delta(m_i)}{\cos m_i - \sigma(m_i)} = \tan\frac{m_i + i\pi}{2}, \quad \text{and}$$

$$\epsilon^{\pm\frac{m_i}{2}} = \sqrt{\sec m_i}\cos\left(\frac{m_i}{2} \mp \left(\frac{i\pi}{2} + \frac{\pi}{4}\right)\right);$$

hence (35) becomes

$$\sqrt{2}\sin\left(\frac{mx}{l} - \frac{m+i\pi}{2}\right) + \epsilon^{\frac{mx}{l}}\cos\left(\frac{m+i\pi}{2} + \frac{\pi}{4}\right)$$
$$- \epsilon^{-\frac{mx}{l}}\sin\left(\frac{m+i\pi}{2} + \frac{\pi}{4}\right) = 0;$$

and, transferring the origin of abscissæ to the middle point of the axis as before, we obtain finally, instead of (34) and (34′), the two equations

$$\sin\left(\frac{m_i x}{l} - \frac{i\pi}{2}\right) + \tfrac{1}{2}\sqrt{\tfrac{1}{2}\sin\beta_i}\,.\left(\epsilon^{\frac{m_i x}{l}} - \epsilon^{-\frac{m_i x}{l}}\cos i\pi\right) = 0; \quad (37)$$

$$\sqrt{2}\sin\left(\frac{m_i x}{l} - \frac{i\pi}{2}\right) + \epsilon^{m_i\left(\frac{x}{l} + \frac{1}{2}\right)}\sin\tfrac{1}{2}\beta_i$$
$$- \epsilon^{-m_i\left(\frac{x}{l} + \frac{1}{2}\right)}\cos i\pi\cos\tfrac{1}{2}\beta_i = 0. \quad (37')$$

200. The equations (34) or (34′) and (37) or (37′) determine, in the two cases, the positions of the nodes for each value of i, that is, for each component tone; the value $i = 1$ belonging to the fundamental tone in both cases. The values of x which lie between $-\frac{1}{2}l$ and $+\frac{1}{2}l$ give the distances of the nodes from the middle point.

The numerical values of m_i in the two cases have already been given. The values of a_i, β_i are the differences (taken positively) between these numbers and the values of $(2i \mp 1)\frac{\pi}{2}$, and are as follows:

$$a_1 = 0.3043, \qquad \beta_1 = 0.0176,$$
$$a_2 = 0.0184, \qquad \beta_2 = 0.0008,$$
$$a_3 = 0.0008, \qquad \beta_3 = 0.0001.$$

(For values of i above 3, there is no significant figure in the first four decimal places.)

201. We will first consider the case of the rod free at both ends, which is the simpler of the two.

It is evident that equation (37) is satisfied by $x = 0$ when i is even; and that, in all cases, if x' be a root, then $-x'$ is a root also.

Hence the nodes are symmetrically distributed with respect to the middle point, as might be foreseen *à priori*; and when i is even, that is, for the 2nd, 4th, &c. component tones, there is a node at the middle point.

For values of i is not greater than 3, the actual numerical values of m_i and β_i must be introduced, and the equation (37) or (37′) solved by approximation.

Since the second term of (37) is essentially positive, the first term must be negative; and from this condition it may be shewn (but more easily by making a graphic construction for one or two particular cases) that the number of roots between $\frac{1}{2}l$ and $-\frac{1}{2}l$, that is, the number of nodes, is $i+1$. Thus the fundamental tone has two nodes, &c. (see Art. **205**).

For greater values of i, it is evident on inspection of the values of β_i given in Art. **200**, that the second term of (37) will be insignificant when $\frac{x}{l}$ is numerically small. Hence, for the higher component tones, and for nodes not near the ends of the rod, the values of x will be such as make the first term vanish, or

$$\frac{m_i x}{l} - i\frac{\pi}{2} = \pm n\pi,$$

n being an integer; and putting in this equation the approximate value $(2i+1)\frac{\pi}{2}$ for m_i, we get

$$\frac{x}{l} = \frac{i \pm 2n}{2i+1}.$$

Thus the nodes which are not near the ends are distributed at sensibly equal distances, the interval between any two consecutive nodes being $\frac{2}{2i+1}l$.

202. But for values of i greater than 3, and for nodes near the ends, we may proceed as follows. The last term of (37′) will be insignificant when x is positive, and $\frac{x}{l}$ not a small fraction. In the second term we may put for $\sin\frac{1}{2}\beta_i$ an approximate value derived from the equation

$$\sigma(m_i) = \sec m_i = \frac{1}{\sin\beta_i}$$

(see Art. **199**), which gives

$$\sin\beta_i = \frac{2}{\epsilon^{m_i}+\epsilon^{-m_i}} = 2\epsilon^{-m_i}$$

(the square of ϵ^{-m_i} being neglected), and therefore

$$\sin \tfrac{1}{2}\beta_i = \epsilon^{-m_i},$$

since β_i is very small. Hence the second term of (37′) becomes

$$\epsilon^{m_i \frac{x}{l} - \frac{m_i}{2}},$$

and if in this term we put the approximate value

$$(2i+1)\frac{\pi}{4} \text{ for } \frac{m_i}{2},$$

the equation becomes

$$\sqrt{2}\sin\left(\frac{m_i x}{l} - \frac{i\pi}{2}\right) + \epsilon^{\frac{m_i x}{l} - \frac{i\pi}{2} - \frac{\pi}{4}} = 0.$$

Now let $\frac{i\pi}{2} - \frac{m_i x}{l} = \vartheta$, then

$$\sqrt{2}\sin\vartheta = \epsilon^{-\vartheta - \frac{\pi}{4}}; \qquad (38)$$

and this equation will have a determinate series of positive roots, say $\vartheta_1, \vartheta_2, \ldots \vartheta_j, \ldots$, which can be found by approximation. The values of x will then be given by the equation

$$m_i \frac{x_j}{l} = \frac{i\pi}{2} - \vartheta_j,$$

or, if m_i be replaced by its approximate value,

$$\frac{x_j}{l} = \frac{i\pi - 2\vartheta_j}{(2i+1)\pi}. \qquad (39)$$

This formula gives the distances, from the middle point, of the nodes towards the positive end. And we know that the nodes in the negative half of the rod are respectively at the same distances from the middle point.

It is easily found (by roughly drawing the two curves

$$y = \sqrt{2}\sin x, \quad \text{and} \quad y = \epsilon^{-x-\frac{\pi}{4}})$$

that, for increasing values of j, ϑ_j tends rapidly to become $(j-1)\pi$; so that the above expression for $\frac{x_j}{l}$ tends to assume the same form as that given in Art. **201** for nodes near the middle. The numerical results will be given below. (Art. **205.**)

203. When one end of the rod is fixed, the nodes are not symmetrically distributed, and the positions of those near the two ends must be found separately. For values of i not greater than 3, the equation (34) or (34′) must be solved by approximation, after the numerical values of a_i have been introduced. But for greater values of i, and for small values of $\frac{x}{l}$ (that is, for nodes not near the ends), we may neglect the second term of (34), and we then find (in the same way as at the end of Art. **201**),

$$\frac{x_j}{l} = \frac{i \pm (2n+1)}{2i-1};$$

from which it follows that, near the middle, the interval between any two consecutive nodes is sensibly equal to $\frac{2l}{2i-1}$, and that when i is odd (being greater than 3) one node (namely, the middle node, if the fixed end be reckoned as one) is sensibly at the middle of the rod.

For nodes near the free end, since $\frac{x}{l}$ is positive, we may neglect the last term in (34′); and in the second term we may introduce the approximate value of a_i derived from the equations (Art. **198**)

$$\sigma(m_i) = -\sec m_i, \qquad \cos m_i = -\sin a_i;$$

which give approximately (as the corresponding equations in Art. **202**)

$$\sin \tfrac{1}{2} a_i = \epsilon^{-m_i};$$

so that (34′) is reduced to

$$\sqrt{2}\cos\left(\frac{m_i x}{l} - \frac{i\pi}{2}\right) + \epsilon^{m_i \frac{x}{l} - \frac{m_i}{2}} = 0;$$

or, since $m_i = (2i-1)\frac{\pi}{2}$ approximately (see (32)),

$$\sqrt{2}\cos\left(\frac{m_i x}{l} - \frac{i\pi}{2}\right) + \epsilon^{m_i \frac{x}{l} - \frac{i\pi}{2} - \frac{\pi}{4}} = 0;$$

so that, if we put

$$\frac{i\pi}{2} - \frac{m_i x}{l} = \theta,$$

we have

$$\sqrt{2}\cos\theta + \epsilon^{-\theta - \frac{\pi}{4}} = 0;$$

and if we call θ, θ_2, ... θ_j, ... the roots of this equation, then

$$m_i \frac{x_j}{l} = \frac{i\pi}{2} - \theta_j, \quad \text{or}$$

$$\frac{x_j}{l} = \frac{i\pi - 2\theta_j}{(2i-1)\pi}; \tag{40}$$

for increasing values of j it is easily seen that θ_j tends to become $(2j-1)\frac{\pi}{2}$.

204. For nodes near the fixed end x is negative, and therefore the second term of (34′), which is (see last Article) approximately

$$\epsilon^{m_i\frac{x}{l}-\frac{m_i}{2}},$$

is small. Neglecting it, and putting 1 for $\cos\frac{a_i}{2}$ in the last term, we have

$$\sqrt{2}\cos\left(\frac{m_i x}{l}-\frac{i\pi}{2}\right)-\epsilon^{-m_i\frac{x}{l}-\frac{m_i}{2}}\cos i\pi = 0;$$

or, since $\frac{m_i}{2} = \frac{i\pi}{2}-\frac{\pi}{4}$ approximately,

$$\sqrt{2}\cos\left(\frac{m_i x}{l}-\frac{i\pi}{2}\right)-\epsilon^{-m_i\frac{x}{l}-\frac{i\pi}{2}+\frac{\pi}{4}}\cos i\pi = 0;$$

and if we now put

$$\frac{m_i x}{l}+\frac{i\pi}{2} = \phi,$$

the equation is reduced to

$$\sqrt{2}\cos\phi-\epsilon^{-\phi+\frac{\pi}{4}} = 0.$$

Let $\phi_1, \phi_2, \ldots \phi_j, \ldots$ be the roots of this last equation; then

$$m_i\frac{x_j}{l} = -\left(\frac{i\pi}{2}-\phi_j\right),$$

$$\text{or}\quad \frac{x_j}{l} = -\frac{i\pi-2\phi_j}{(2i-1)\pi}; \tag{41}$$

while, for increasing values of j, ϕ_j tends to become $(2j+1)\pi$.

205. The following numerical results have been given by Seebeck[1], who, however, has treated the fundamental equations (30) and (35) in a somewhat different manner.

Case I. (One end fixed and the other free.)

[1] In a memoir on the transverse vibrations of rods. (*Abhandlungen d. Math. Phys. Classe d. K. Sächs. Gesellschaft d. Wissenschaften.* Leipzig, 1852.)

Distances of nodes from the free end, the length of the rod being taken as unity:

2nd tone,	0.2261,					
3rd	0.1321,	0.4999,				
4th	0.0944,	0.3558,	0.6439,			
ith	$\frac{1.3222}{4i-2}$,	$\frac{4.9820}{4i-2}$,	$\frac{9.0007}{4i-2}$,	$\frac{4j-3}{4i-2}$,	$\frac{4i-10.9993}{4i-2}$,	$\frac{4i-7.0175}{4i-2}$.

The last row in this table must be understood as meaning that $\frac{4j-3}{4i-2}$ may be taken as the distance of the jth node from the free end, except for the first three and last two nodes.

Case II. (Both ends free.)

Distances of nodes from nearest end:

1st tone,	0.2242,			
2nd	0.1321,	0.5,		
3rd	0.0944,	0.3558,		
ith	$\frac{1.3222}{4i+2}$,	$\frac{4.9820}{4i+2}$,	$\frac{9.0007}{4i+2}$,	$\frac{4j-3}{4i+2}$.

206. To complete the theory of the transverse vibrations of a rod, it is necessary to shew how the form of the axis, at any time during the motion, is determined by the initial displacements and velocities of its points.

On reference to Art. **179**, &c. it will be seen that in all the cases which have been considered, the equation to the axis at the time t is of the form

$$y = \Sigma_{i=1}^{i=\infty} u_i\,(A_i \cos n_i t + B_i \sin n_i t)\,; \qquad (42)$$

in which $n_1, n_2, \ldots n_i, \ldots$ are determinate constants, depending upon the roots of an equation, in general transcendental, and u_i is a determinate function of x and of n_i, which satisfies a differential equation such as (8), in which the coefficients depend (through m) upon n_i. We shall consider only the case in which there is no tension, so that $a = 0$, and shall also neglect, as before, the term $m^2 \frac{d^2u}{dx^2}$, which arises from the term $\frac{d^4y}{dt^2\,dx^2}$ in

(6′), introduced by taking account of the angular motion of the transverse sections of the rod.

With this simplification we may write equation (8) thus for any two different roots of the transcendental equation referred to above:

$$\frac{d^4 u_i}{dx^4} + p_i u_i = 0,$$

$$\frac{d^4 u_j}{dx^4} + p_j u_j = 0,$$

in which p_i, p_j are two constants, of which we only require to know that they are different.

From these equations we have

$$u_j \frac{d^4 u_i}{dx^4} - u_i \frac{d^4 u_j}{dx^4} + (p_i - p_j) u_i u_j = 0.$$

Now if we multiply this by dx, and integrate from $x = 0$ to $x = l$, the result is

$$(p_i - p_j) \int_0^l u_i u_j \, dx = 0;$$

for the first two terms of the equation, multiplied by dx, are the differential of

$$u \frac{d^3 u_i}{dx^3} - u_i \frac{d^3 u_j}{dx^3} + u_i \frac{d^2 u_j}{dx^2} - u_j \frac{d^2 u_i}{dx^2},$$

of which every term vanishes at both limits, on every supposition as to the terminal conditions. (See Art. **187.**)

It follows therefore that when j is different from i

$$\int_0^l u_i u_j \, dx = 0, \tag{43}$$

but $\int_0^l u_j^2 \, dx$

will be a determinate constant, depending upon j.

Now the initial displacements and velocities being supposed to be given, we have, when $t = 0$,

$$y = f(x), \qquad \frac{dy}{dt} = \phi(x),$$

where $f(x)$ and $\phi(x)$ are functions of which the value is given for all values of x from 0 to l. Hence, from (42),

$$\Sigma A_i u_i = f(x), \qquad \Sigma n_i B_i u_i = \phi(x);$$

and if these equations be multipled by $u_j dx$, and integrated from $x = 0$ to $x = l$, the result (by (43)) is

$$A_j \int_0^l u_j^2\, dx = \int_0^l f(x)\, u_j\, dx,$$

$$n_j B_j \int_0^l u_j^2\, dx = \int_0^l \phi(x)\, u_j\, dx;$$

so that A_j, B_j are determined, and the form of the axis of the rod at any time t is then given by equation (42).

Torsion Vibrations.

207. Torsion vibrations may be properly included in the general class of lateral vibrations; but as they are of little practical importance we shall discuss them briefly. Such vibrations will be produced when a uniform elastic rod is left to itself after undergoing a slight disturbance by forces reducible to couples in planes perpendicular to its axis.

If the rod were in equilibrio under the action of such forces, it would be in a state of torsion or twist; and the twist may be called *simple* when the particles in any transverse plane section are not displaced relatively to one another, and the distance between any two sections remains unaltered.

Suppose a cylindrical rod, whether solid or hollow (as a tube), to be twisted by equal and opposite couples applied only in the planes of its ends; then, if there is no relative displacement of particles in either of these terminal sections, it is evident that there will be none in any other transverse section; and it is known that the length of the rod remains unaltered, or rather is altered only by a quantity of the second order, when the twist is small. Under the action of such forces the rod, when in equilibrio, will be in a state of uniform simple twist. It is probable that such a condition cannot be realised in practice except in the case of cylindrical rods, though it may subsist, more or less approximately, for other forms. In what follows we shall assume that the form is cylindrical.

208. When the twist is uniform, *the rate of twist* is defined by the angle through which any transverse section is turned relatively to any other, divided by the distance between the two sections. And the limit of this ratio, when the distance between the two sections is diminished indefinitely, is the rate of twist in that section with which they ultimately coincide, whether the twist be uniform or not.

When the twist is uniform, all the particles which, in the untwisted state, lay upon any straight line parallel to the axis, will lie upon a helix. And the inclination of a tangent to this helix, at any point, to the axis, will be directly proportional to the rate of twist and to the distance from the axis. When this inclination is small where it has its greatest value, that is, on the exterior surface, the twist may be called small. It is evident that a small twist is consistent with a large relative angular displacement of the terminal sections, if the radius of the cylinder be small compared with its length.

209. When equilibrium subsists under the action of couples applied only in the planes of the terminal sections, it is evident that the moments of these couples must be equal and opposite; and it is known from experiment that when the twist is small, the rate of twist is, for a rod of given material and section, proportional to the moment of the couples.

If, besides the terminal couples, there are twisting forces acting on the interior matter of the rod, the conditions of equilibrium are easily found as follows. Let x be the distance of any transverse section from one end (A) of the rod, and θ the angular displacement of that section. Then $\frac{d\theta}{dx}$ (Art. **208**) is the rate of twist in a section at the distance x from A; and the moment of the couple which would have to be applied in the plane of that section in order to maintain equilibrium, if the rod were cut there, would be $C\frac{d\theta}{dx}$, C being a constant which we shall consider more particularly below.

210. Let us consider then an infinitesimal slice contained between two sections at distances $x-\frac{1}{2}dx$, $x+\frac{1}{2}dx$ from A. If all the rest of the rod were removed, it would be necessary, in order to maintain the equilibrium of the slice, to apply in its two faces couples of which the moments are

$$-C\left(\frac{d\theta}{dx}-\frac{1}{2}\frac{d^2\theta}{dx^2}dx\right), \quad C\left(\frac{d\theta}{dx}+\frac{1}{2}\frac{d^2\theta}{dx^2}dx\right)$$

(those couples being considered positive which tend to increase θ).

Hence, if $L\rho\omega\,dx$ be the moment of the twisting forces acting on the mass of the slice, where ρ, ω are the density, and area of section of the rod, the condition of equilibrium will be

$$L\rho\omega\,dx + C\frac{d^2\theta}{dx^2}dx = 0,$$

and the differential equation of motion will be obtained from this as usual, by substituting for $L \rho \omega dx$ the sum of the moments of the resistances to acceleration of the particles of the slice, namely,

$$-\rho\, dx \frac{d^2\theta}{dt^2} \int r^2\, d\omega,$$

where $d\omega$ is an element of area of the transverse section, and r the distance of $d\omega$ from the axis. If then we put k for the radius of gyration of the area of the section about the axis of the rod, so that

$$\int r^2\, d\omega = k^2 \omega,$$

we shall have

$$\frac{d^2\theta}{dt^2} = \frac{C}{k^2 \rho \omega} \frac{d^2\theta}{dx^2} \qquad (1)$$

for the equation of motion to be satisfied at all parts of the rod.

The terminal conditions will be

$$\theta = 0 \text{ at a fixed end, and}$$

$$\frac{d\theta}{dx} = 0 \text{ at a free end.}$$

(The latter condition is evident if it be observed that at a free end the rate of twist must be o, since there is no couple in the terminal face.)

211. On the supposition that the material of the rod is *isotropic* (*Tait and Thomson*, § 676), and therefore equally elastic in all directions, the constant C can be expressed in terms of q, the modulus of elasticity (Art. **148**), and of another constant μ, the meaning of which we will now explain.

If a uniform bar, of any section, be extended by forces applied uniformly to the surfaces of its ends only, it is known that the transverse linear dimensions are contracted. Let ϵ be the longitudinal extension (see Art. **148**), and $\mu\epsilon$ the transverse or lateral contraction; then, the extensions and contractions being always supposed small, μ is a constant for a given material, and moreover must have a value between o and $\frac{1}{2}$, if, as is the case with all ordinary substances, the *volume* of the bar is increased under the circumstances supposed.

It can be shewn that in the case of a cylindrical, solid or hollow, rod, the value of the constant C in the last Article is[1]

$$C = \frac{k^2 \omega q}{2(1+\mu)};$$

[1] The demonstration of these propositions is elementary, but we cannot afford space for it here. See *Klebsch*, §§ 2, 3, 92.

so that equation (1) becomes

$$\frac{d^2\theta}{dt^2} = \frac{q}{2(1+\mu)\rho}\,\frac{d^2\theta}{dx^2}. \qquad (2)$$

212. We need not repeat the process of integrating (2), since it is exactly analogous to that which has been applied in former problems (see Art. **122**) and can offer no difficulty. We shall merely give the result in two cases.

(1) If both ends of the rod are free, then

$$\theta = \sum_{i=1}^{i=\infty} A_i \cos\frac{i\pi x}{l}\,\sin\left(\frac{i\pi a t}{l} + a_i\right).$$

(2) If the end from which x is measured is fixed, and the other end free, then

$$\theta = \sum_{i=0}^{i=\infty} A_i \sin\frac{(2i+1)\pi x}{2l}\,\sin\left(\frac{(2i+1)\pi a t}{2l} + a_i\right);$$

where A_i, a_i, in each case, represent arbitrary constants, to be determined by initial circumstances, and a is defined by the equation

$$a^2 = \frac{q}{2(1+\mu)\rho}.$$

The period of the i^{th} tone is therefore

$$\frac{2l}{i}\left(\frac{2(1+\mu)\rho}{q}\right)^{\frac{1}{2}} \quad \text{in case (1)},$$

$$\frac{4l}{2i+1}\left(\frac{2(1+\mu)\rho}{q}\right)^{\frac{1}{2}} \quad \text{in case (2)}.$$

Now comparing these with the periods of longitudinal vibrations of the same rod under the same terminal conditions, we see that the tone of given order produced by torsion vibrations is lower than that of the same order produced by longitudinal vibrations, by an interval of which the ratio is

$$(2(1+\mu))^{\frac{1}{2}}.$$

213. The value of the constant μ is probably different for different substances. Navier and Poisson, by reasoning now generally admitted to be illegitimate, deduced *a priori* the value $\mu = \frac{1}{4}$ for all substances.

Wertheim found experimentally[1] $\mu = \frac{1}{3}$ for glass and brass.

Kirchoff[2] found values differing sensibly from this for steel bars, and for a drawn brass bar in which the longitudinal elasticity differed from the lateral.

[1] *Ann. de Chim. et de Phys.* 3rd series, vol. xxiii. p. 54.
[2] *Poggendorf*, vol. cviii. p. 369.

The results of the last Article would afford an experimental means of determining μ, if it were possible to be assured that the rods used were isotropic, and to observe with sufficient precision the intervals between the tones given by longitudinal and torsion vibrations.

Chladni asserts that this interval is always a *fifth*. If this were so, or rather, for substances in which it is so, we must have $2(1+\mu) = \frac{9}{4}$, or $\mu = \frac{1}{8}$. If the value of μ were $\frac{1}{3}$, the ratio of the interval would be $(\frac{8}{3})^{\frac{1}{2}} = 1.632$.

It is impossible however, or at any rate very difficult, to observe with great exactness the interval between the two tones, and a small error in the ratio of the interval may evidently produce a considerable error in the value of μ. Hence this constant must be determined by other methods.

214. Torsion vibrations may be excited in a cylindrical rod by friction with the same substances as would excite longitudinal vibrations in the same rod.

Thus, if a piece of stout glass tube, four or five feet long, be gently but firmly clamped in a table-vice at its middle, after winding a piece of broad tape about it at that part to protect it from the vice, and if a wet piece of the same tape be passed once round the tube not far from its middle, and the ends rather lightly and quickly pulled backwards and forwards at right angles to the tube by the two hands, the torsion vibrations will be easily produced.

When the rod is not cylindrical the friction of a bow (charged as usual with powdered rosin) should be used. Thus the torsion vibrations of a rectangular deal rod may be excited by clamping one end in a vice and drawing the bow across one of its edges at right angles to the rod, at a point distant from the fixed end about a fourth of the length.